

How to initialize a second class particle?

Márton Balázs*

m.balazs@bristol.ac.uk

Attila László Nagy†

attilalaszlo.nagy@gmail.com

January 15, 2016

Abstract

We greatly generalize P. A. Ferrari and C. Kipnis' [17] results on the behavior of the second class particle in the rarefaction fan of the totally asymmetric simple exclusion process. Versions of their results are shown to hold through for practically any attractive particle system (including zero-range, misanthrope models, and many more) with established hydrodynamic behavior. The main novelty is the introduction of a *signed* coupling measure as initial data, which nevertheless results in a proper probability initial distribution for the site of the second class particle. This distribution proves to be canonical in many senses and makes the extension of [17] possible. Combined with strong recent results in hydrodynamic limits, we are able to identify the ballistically and diffusively rescaled limit distribution of the second class particle position in a wide range of asymmetric and symmetric models, respectively. We also point out a model with non-concave, non-convex hydrodynamics, where the rescaled second class particle distribution has both continuous and discrete counterparts. As a by-product of our methods we reveal a very interesting invariance property of the one-site marginal distribution of the process underneath the second class particle. Finally, we give a lower estimate on the probability of survival of a second class particle-antiparticle pair.

Keywords. second class particle, limit distribution, rarefaction fan, shock, hydrodynamic limit, collision probability.

Acknowledgement. The authors thank valuable discussions with Pablo A. Ferrari on the problem, with Ellen Saada on the hydrodynamic limit of asymmetric processes and with Cédric Bernardin on symmetric processes. M. Balázs acknowledges support from the Hungarian Research Funds (OTKA) K100473 and K109684.

1 Introduction

This paper studies the behavior of second class particles in a wide class of one-dimensional attractive particle systems. The evolution of such particles can be obtained by coupling two systems (of first class particles) coordinate-wise in such a manner that their initial configurations only differ at finitely many places. Second class particles interact with the underlying process and perform highly nontrivial motion which is only partially understood in general. In asymmetric models they are known, in first order, to follow the characteristic lines of the limiting hydrodynamic equation of the density. In three classical cases: translation-invariant stationary, rarefaction fan, and shock scenario this results in a law of large numbers with the characteristic velocity, a random admissible characteristic velocity of the rarefaction fan, and the speed of the shock, respectively. These make the second class particle a relevant microscopic object that captures macroscopic properties of the ambient system. Fluctuations show superdiffusive scaling for translation-invariant stationary, rarefaction fan, deterministic shock initial data and diffusive scaling for random shock initial data. Many of the previous properties have been proven rigorously for the most-studied totally asymmetric simple exclusion process (TASEP) and in some cases for other processes as well.

*School of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, United Kingdom.

†Institute of Mathematics, Budapest University of Technology and Economics, Egrý J. u. 1., Budapest, H-1111, Hungary.

However, they are conjectured to hold in a wide range of particle systems. Second class particles in symmetric systems have not been much explored, in some simple cases diffusive behavior is known.

This paper makes use of a translation argument to investigate the second class particles under *rarefaction fan* and *shock* initial conditions. We build on the seminal paper [17] by P. A. Ferrari and C. Kipnis on this problem for the TASEP. Their argument compares a step initial product Bernoulli distribution with its translated version and notices that the joint realization of these two can be understood as a coupled initial distribution with possibly a second class particle at the origin. The second class particle of simple exclusion is a *unique object* in the sense that it can *only* couple a process of *zero* particles with one of *one* particle at the site of the second class particle. Together with the natural joint dominant realization of two Bernoulli distributions of different parameters, this makes the above translation argument rather transparent. Naturally, it also follows that the second class particle always has 0 or 1 particles on its site depending on which of the two coupled exclusions we look at. When dealing with systems of more choices for one site occupation numbers, the second class particle stops being a uniquely determined object. Stochastic domination of the natural measures associated with attractive models still holds, but the actual realization of a coupled pair has some details to fix besides its marginals. In particular, it is not clear whether two models with slightly different densities can be coupled using zero or one second class particles per site only, or more than one of them on a site have to be assumed with positive probability. Actually, this latter is the case for popular stationary distributions as the ones of Geometric or Poisson marginals (e.g., for zero-range processes).

We build up a natural initial distribution for the second class particle in step initial configurations which allows for an extension of P. A. Ferrari and C. Kipnis' arguments. We can do this even when coupling with zero or one second class particles only fails. This is where the main novelty of the paper lies: to force zero or one second class particles with the correct one-site marginals for the coupled pair, one has to introduce negative weights in the coupling measure. As it turns out negative weights only belong to configurations without a second class particle, and this non-physical coupling measure always assigns positive weights to states with a second class particle. These can then be normalized to a proper probability distribution that a.s. has the second class particle.

This construction provides a canonical initial distribution to start a second class particle from. Under this initial distribution we connect the probabilities of the second class particle position to other, easier quantities of a model without second class particles. Together with recent results of hydrodynamics we can then proceed to prove limit distribution results on the rescaled position of the second class particle. Both asymmetric and symmetric systems are handled under the natural scaling that fits the respective scenario. The limit distributions then relate to the solution of the hydrodynamic equation with step initial condition. There are two particular and interesting instances, to the best of our knowledge not much explored in the literature, of second class particle-behavior:

- (i) in asymmetric models with non-concave and non-convex hydrodynamic flux, shocks and rarefaction fans can coexist and the limit distribution of the second class particle reflects this fact by developing both continuous and discrete components at the same time; and
- (ii) central limit theorem for the second class particle is pointed out in a symmetric system where, as opposed to simple symmetric exclusion, it is *not* a simple random walker.

As a by-product of our arguments we are able to relate the one-site marginal of the first class particles at the site of the second class particle to the distribution of a model without the second class particle. Under certain initial distributions this results in a time-stationary one-site marginal – a quite unexpected result. Finally, we push the arguments, in line with [17], to give a lower estimate on the survival probability of a second class particle-antiparticle pair in general models.

Earlier results. A review and several open problems appeared in [15, 22] many of which are completely solved by the present paper. A law of large numbers for the position of the second class particle of exclusion and zero range processes with shock initial condition was obtained by F. Rezakhanlou [31]. Note that his initial setup of the second class particle slightly differs from ours. As described above, in case of the rarefaction fan (and for the TASEP) the first and

fundamental paper was [17]. T. Mountford and H. Guiol [28] then sharpened [17] by proving that the convergence takes place almost surely. Recently P. Gonçalves has translated the results of [17] for the totally asymmetric constant rate zero-range process in [21] via a direct coupling between exclusion and zero-range. P. A. Ferrari, P. Gonçalves and J. B. Martin [16] have very elegant arguments on collision probabilities in exclusion processes. Many results on the behavior of the second class particle in the TASEP have been reproven by P. A. Ferrari and L. P. R. Pimentel [19] and by P. A. Ferrari, J. B. Martin and L. P. R. Pimentel [18], translating the problem into one of competition interfaces in last passage percolation. D. Romik and P. Śniady [32] pointed out a very elegant algebraic connection between the motion of second class particles in a variant of the TASEP and an evolution, so-called “jeu de taquin”, defined on infinite Young tableaux through which the distributional limit was proved. A generalization of the results of [17] for TASEP equipped with *higher order* particles (like third, fourth, etc. class particles), known as *multi-type* TASEP, was discussed in [1] by G. Amir, O. Angel and B. Valkó. Using a powerful analytic approach, exact expressions were obtained by C. A. Tracy and H. Widom [36] for the second class particle evolving in the asymmetric simple exclusion process.

Organization of the paper. We give a precise definition of the processes and the corresponding notions in Sections 2, 3 and 4. Section 5 contains the main results while the proofs are postponed to Section 7. We outline and discuss several examples of models in Section 6.

Notations. As usual, \mathbb{R} , \mathbb{Z} and \mathbb{N} are the set of reals, integers and naturals (positive integers), respectively, while \mathbb{R}_0^+ (\mathbb{Z}_0^+) denotes the non-negative reals (integers). Throughout the article $\mathbb{1}\{\cdot\}$ stands for the indicator function.

2 Models

The model class we investigate originates in the work of Coccozza-Thivent [13], extensions and several examples we cover first appeared in the papers [6, 35]. We consider general, nearest neighbor stochastic interacting particle systems

$$\omega := (\omega(t))_{t \geq 0} = ((\omega_i(t))_{i \in \mathbb{Z}})_{t \geq 0}$$

on the configuration space $\Omega := \mathcal{I}^{\mathbb{Z}}$ with

$$\mathcal{I} = \{\omega^{\min}, \omega^{\min} + 1, \dots, \omega^{\max} - 1, \omega^{\max}\} \subset \mathbb{Z}$$

such that $-\infty \leq \omega^{\min} < \omega^{\max} \leq +\infty$. In particular \mathcal{I} can as well be an infinite subset of \mathbb{Z} . The quantity $\omega_i(t)$ denotes the number of (signed) particles sitting on the i^{th} lattice point at time $t \in \mathbb{R}_0^+$. We adopt this interpretation even if $\omega_i(t)$ happens to be negative.

The dynamics we attach on top of the configuration space Ω is a continuous time Markov jump dynamics that allows the particles to execute right as well as left jumps with respective instantaneous rates r and ℓ . Formally, with the *Kronecker symbol*

$$(\delta_i)_j := \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{otherwise,} \end{cases}$$

the transitions are of the form

$$\omega \xrightarrow{r_i(\omega)} \omega - \delta_i + \delta_{i+1} \in \Omega, \quad \omega \xrightarrow{\ell_i(\omega)} \omega + \delta_i - \delta_{i+1} \in \Omega. \quad (2.1)$$

Conditioned on a given configuration, these steps take place independently for each $i \in \mathbb{Z}$ with the above respective rates. We pose the following assumptions for the rates throughout the article.

Translation invariance. For each fixed $i \in \mathbb{Z}$ and $n \in \mathbb{Z}$:

$$r_i(\omega) = r_{i+n}(\bar{\omega}) \quad \text{and} \quad \ell_i(\omega) = \ell_{i+n}(\bar{\omega}),$$

whenever $\bar{\omega}_j = \omega_{j+n}$ for every $j \in \mathbb{Z}$.

Finite range dependence. There exists a constant $K \in \mathbb{Z}^+$ such that for every $i \in \mathbb{Z}$ we have

$$r_i(\omega) = r_i(\bar{\omega}) \quad \text{and} \quad \ell_i(\omega) = \ell_i(\bar{\omega})$$

whenever $\omega_j = \bar{\omega}_j$ holds for all j with $|i - j| \leq K$.

Non-degeneracy. For every $i \in \mathbb{Z}$: $r_i(\omega) = 0$ ($\ell_i(\omega) = 0$) if and only if $\omega_i = \omega^{\min}$ ($\omega_{i+1} = \omega^{\min}$) or $\omega_{i+1} = \omega^{\max}$ ($\omega_i = \omega^{\max}$) hold, whenever $-\infty < \omega^{\min}$ or $\omega^{\max} < \infty$, respectively.

In particular the third assumption will make sure that the process a.s. keeps the state space Ω . Sometimes we will let one of the left or right jump rates be zero (totally asymmetric case). The (formal) *infinitesimal generator* \mathcal{G} of our Markov process acts on a *cylinder function* (one that depends only on a finite number of coordinates of $\omega \in \Omega$) $\varphi : \Omega \rightarrow \mathbb{R}$ as

$$\begin{aligned} (\mathcal{G}\varphi)(\omega) &= \sum_{j \in \mathbb{Z}} r_j(\omega) \cdot (\varphi(\omega - \delta_j + \delta_{j+1}) - \varphi(\omega)) \\ &\quad + \sum_{j \in \mathbb{Z}} \ell_j(\omega) \cdot (\varphi(\omega + \delta_j - \delta_{j+1}) - \varphi(\omega)). \end{aligned} \tag{2.2}$$

If $\sup_{j \in \mathbb{Z}, \omega \in \Omega} r_j(\omega)$ and $\sup_{j \in \mathbb{Z}, \omega \in \Omega} \ell_j(\omega)$ are both finite then the above Markov process can be constructed on Ω in an appropriate manner having generator \mathcal{G} (see [27, Chapter 1]). In other cases, existence of the dynamics can only be established by posing further (growth) conditions on the rates (see [2], [10] and further references therein). Within the scope of this article we do not intend to deal with this issue in general, though we will discuss some models with unbounded rates in Section 6. From now on we assume that the processes can be constructed with appropriate initial data in Ω with the above dynamics.

In the next section we introduce the *attractivity condition*, which will further tighten model class.

3 Second class particles

Our main object of investigation will be the second class particle. In order to define the notion we first pick two configurations: $\omega, \eta \in \Omega$ aligning them coordinate-wise. We can then define the *number* $n_i = |\omega_i - \eta_i|$ and the *sign* $s_i = \mathbb{1}\{\omega_i - \eta_i > 0\} - \mathbb{1}\{\omega_i - \eta_i < 0\}$ of *signed second class particles* at position $i \in \mathbb{Z}$ in the configuration pair (ω, η) . In particular, if

$$\omega = \eta + \delta_0 \quad (\omega = \eta - \delta_0), \tag{3.1}$$

then we say that a single positive (negative) second class particle is placed at the origin in (ω, η) . To allow second class particles evolve in time we use the *basic*, “*particle-to-particle*”, *coupling*, that is for each time instance $t > 0$ and lattice point $i \in \mathbb{Z}$, a hop to the right can occur in both systems:

$$(\omega, \eta) \longrightarrow (\omega - \delta_i + \delta_{i+1}, \eta - \delta_i + \delta_{i+1}) \in \Omega \times \Omega$$

with rate $\min(r_i(\omega), r_i(\eta))$; while “compensating” right jumps occur according to the following rules with respective rates:

$$\frac{\left| \begin{array}{c} (\omega - \delta_i + \delta_{i+1}, \eta) \\ (\omega, \eta) \end{array} \right|}{(r_i(\omega) - r_i(\eta))^+} \left| \begin{array}{c} (\omega, \eta - \delta_i + \delta_{i+1}) \\ (\omega, \eta) \end{array} \right|}{(r_i(\omega) - r_i(\eta))^-}.$$

Here $(\cdot)^+$ and $(\cdot)^-$ denote the positive and negative part function, respectively. The coupling tables for the left jumps can be obtained analogously. Note that a second class particle can hop only if a compensating step occurs. Also notice that under the basic coupling the marginal processes, that is $(\omega(t))_{t \geq 0}$ and $(\eta(t))_{t \geq 0}$, follow the same stochastic evolution rules (2.1). Now, recall the following notion from [27, Definition 2.3, pp. 72].

Definition 1. We say that the dynamics defined by the infinitesimal generator \mathcal{G} of (2.2) is *attractive*, if the initial dominance $\eta(0) \leq \omega(0)$ implies the one $\eta(t) \leq \omega(t)$ for all times $t > 0$ under the basic coupling.

From now on we will always assume that the rates r and ℓ are chosen in a manner that makes the dynamics attractive. In these processes, the above (basic) coupling tables reveal some extra properties for second class particles. In particular, having initial configurations as in (3.1), a.s. there will always be a single second class particle in the system, the position of which will be denoted by $Q(t)$ at time t . More generally, one can see that the total number $\sum_{j \in \mathbb{Z}} |\omega_j(t) - \eta_j(t)| = \sum_{j \in \mathbb{Z}} n_j$ of (positive as well as negative) second class particles is non-increasing in time. That is they cannot give birth to other second class particles, but two of distinct charges can instantaneously *annihilate* each other upon meeting.

4 The initial distribution

The crucial point for our results turns out to be choosing the initial measure on $\Omega \times \Omega$ appropriately. We devote a full section to discuss the choice we make. In this direction we formulate our fairly general assumption below.

Assumption 1. Let $\nu := (\nu^\varrho)_{\varrho \in \mathcal{D}}$ be a family of probability measures on \mathcal{I} , where \mathcal{D} is a bounded subset of $[\omega^{\min}, \omega^{\max}]$ that satisfies the following properties:

- it is parameterized by its mean, that is $\varrho = \sum_{y \in \mathcal{I}} y \cdot \nu^\varrho(\{y\})$ holds for every $\varrho \in \mathcal{D}$; and
- for each $\varrho > \lambda$, where $\varrho, \lambda \in \mathcal{D}$, the measure ν^ϱ stochastically dominates ν^λ , that is $\nu^\lambda(\{z : z \leq y\}) \geq \nu^\varrho(\{z : z \leq y\})$ holds for every $y \in \mathcal{I}$.

In the sequel, we will refer to \mathcal{D} as the *set of densities*. Now, fixing two densities $\varrho > \lambda$ of \mathcal{D} and assuming the above properties for ν , we define the measure $\hat{\nu}^{\varrho, \lambda}$ on $\mathcal{I} \times \mathcal{I}$ as

$$\hat{\nu}^{\varrho, \lambda}(x, y) = \frac{1}{\varrho - \lambda} (\nu^\lambda(\{z : z \leq y\}) - \nu^\varrho(\{z : z \leq y\})) \cdot \mathbf{1}\{x = y + 1\}, \quad (4.1)$$

where $x, y \in \mathcal{I}$. It is an easy exercise to check that this indeed defines a probability distribution. Notice that $\omega_0 = \eta_0 + 1$ holds $\hat{\nu}^{\varrho, \lambda}$ -a.s. By a slight abuse of notation we also set

$$\nu^{\varrho, \varrho}(x, y) := \nu^\varrho(x) \cdot \mathbf{1}\{x = y\}, \quad \text{and} \quad \nu^{\lambda, \lambda}(x, y) := \nu^\lambda(x) \cdot \mathbf{1}\{x = y\} \quad (4.2)$$

as diagonal measures on $\mathcal{I} \times \mathcal{I}$. Now, we are ready to define the initial probability distribution as a site-wise product coupling measure on the space $\Omega \times \Omega$:

$$\hat{\mu}^{\varrho, \lambda} := \bigotimes_{i=-\infty}^{-1} \nu^{\varrho, \varrho} \otimes \hat{\nu}^{\varrho, \lambda} \otimes \bigotimes_{i=1}^{\infty} \nu^{\lambda, \lambda}. \quad (4.3)$$

We start a coupled pair of processes under the initial distribution $\hat{\mu}^{\varrho, \lambda}$, and we denote the associated probability and expectation by $\hat{\mathbf{P}}$ and $\hat{\mathbf{E}}$, respectively. Notice that $\hat{\mathbf{P}}$ a.s. has a second class particle that initially starts from the origin.

Next, we highlight some further properties of the measure (4.3). We will also see why it serves as a natural choice for initial distribution.

Proposition 1. Suppose that Assumption 1 holds and let $\varrho - 1 \leq \lambda < \varrho$, where $\varrho, \lambda \in \mathcal{D}$ are fixed. Then there exists a joint probability measure $\nu^{\varrho, \lambda}$ with ν^ϱ and ν^λ as respective marginals and with $\nu^{\varrho, \lambda}(\{(x, y) : x - y \in \{0, 1\}\}) = 1$ if and only if

$$\nu^\varrho(\{z : z \leq y\}) \geq \nu^\lambda(\{z : z \leq y - 1\}) \quad (4.4)$$

holds for every $y \in \mathcal{I}$. In this case $\hat{\nu}^{\varrho, \lambda}$ can be obtained as

$$\hat{\nu}^{\varrho, \lambda}(\cdot) = \nu^{\varrho, \lambda}(\cdot \mid \omega_0 = \eta_0 + 1) = \frac{\nu^{\varrho, \lambda}(\cdot \cap \{(x, y) : x = y + 1\})}{\nu^{\varrho, \lambda}(\{(x, y) : x = y + 1\})}, \quad (4.5)$$

where $0 < \nu^{\varrho, \lambda}(\{(x, y) : x = y + 1\}) = \varrho - \lambda \leq 1$.

Now, under the narrower assumptions of Proposition 1, we can set up another measure, namely

$$\mu^{\varrho, \lambda} := \bigotimes_{i=-\infty}^{-1} \nu^{\varrho, \varrho} \otimes \nu^{\varrho, \lambda} \otimes \bigotimes_{i=1}^{\infty} \nu^{\lambda, \lambda},$$

which we can call the *unconditional version* of $\hat{\mu}^{\varrho, \lambda}$, since this latter can be obtained from $\mu^{\varrho, \lambda}$ by conditioning on the existence of a single second class particle at the origin. When it does not cause any confusion we will always refer to $\hat{\mu}$ as the “conditional measure”.

Some, but not all, interacting particle systems have translation-invariant product stationary distributions. For those with product measures, it seems natural to choose the marginals ν^{ϱ} and ν^{λ} to be these stationary marginals. As two classical examples, the product of Geometric and Poisson distributions on \mathbb{Z}_0^+ are stationary for zero-range processes with constant and linear rate functions, respectively, to be discussed in Section 6 in more details. Notice, as the following Proposition 2 also demonstrates, that the additional requirement (4.4) of Proposition 1 might be too restrictive in some cases where $\nu^{\varrho, \lambda}$, hence $\mu^{\varrho, \lambda}$, might *not* exist as a *probability* measure.

Proposition 2. *The family of Geometric as well as Poisson distributions can be parameterized to fulfill Assumption 1 but there do not exist different densities ϱ, λ for which (4.4) would hold for every $x \geq 0$ simultaneously.*

Nevertheless, our main result (see Theorem 1 of Section 5) and our techniques do *not* require the existence of the measure $\mu^{\varrho, \lambda}$, in particular that of $\nu^{\varrho, \lambda}$. That is, in the following, we do not assume (4.4) to hold.

In fact we do not even need to start with stationary marginals. In the second part of Subsection 5.1 below we will show a surprisingly large class of initial distributions, far from being stationary in general, where our results also apply.

5 Main results

The results are divided into four subsections below concerning the distribution of the position of a single second class particle in finite time (Subsection 5.1), its large scale limit distributions in the asymmetric as well as in the symmetric case (Subsection 5.2 and Subsection 5.3, respectively), and the collision probability for two second class particles (Subsection 5.4).

First, with $\varrho, \lambda \in \mathcal{D}$, we define $\mathbf{E}_{\sigma^{\varrho, \lambda}}$ to be the expectation according to the law of the process that starts off from the product initial distribution

$$\sigma^{\varrho, \lambda} := \bigotimes_{i=-\infty}^0 \nu^{\varrho} \otimes \bigotimes_{i=1}^{+\infty} \nu^{\lambda}, \quad (5.1)$$

and the evolution of which is described by the infinitesimal generator \mathcal{G} of (2.2). Note that the measure $\sigma^{\varrho, \lambda}$ describes a particular marginal of density ϱ for each site on the left and another marginal of density λ for sites strictly on the right of the origin. Hence whenever $\varrho \neq \lambda$ it is also called as the microscopic *Riemannian density profile* (see also (5.8)) or simply the step initial condition.

We introduce one more notation which will be helpful in the following. For a fixed $n \in \mathbb{Z}$ denote by τ_n the *shift operator* which acts on a configuration $\omega \in \Omega$ as $(\tau_n \omega)(i) = \omega_{i+n}$ ($i \in \mathbb{Z}$) and on a measure $\kappa : \Omega \rightarrow [0, 1]$ as $\tau_n \kappa = \kappa(\tau_n \omega)$, respectively.

5.1 Distribution of the second class particle

The first result connects the law of the displacement of a single second class particle with that of a (first class) particle occupation variable. Fix $\varrho > \lambda$, and recall that, under Assumption 1, $\hat{\mathbf{P}}$ denotes the probability of the coupled process $(\hat{\omega}, \hat{\eta})$ started from product distribution $\hat{\mu}^{\varrho, \lambda}$ of (4.3), while $Q(t)$ is the position of the single second class particle at time t .

Theorem 1 (Displacement distribution of the second class particle). *Suppose that a family of measures ν fulfills Assumption 1. Then for any $n \in \mathbb{Z}$ and $t \in \mathbb{R}_0^+$ we have*

$$\hat{\mathbf{P}}\{Q(t) \leq n\} = \frac{\varrho - \mathbf{E}_{\sigma^{\varrho, \lambda}} \omega_{n+1}(t)}{\varrho - \lambda}. \quad (5.2)$$

Remark 1. Note that ν does not have to be related to the stationary distributions of ω or of $(\hat{\omega}, \hat{\eta})$ in any way. Also notice that we had no further assumptions on the rates r and ℓ , hence *both* asymmetric and symmetric processes are included in the above assertion. Indeed, a careful overview of our technique (see the proof of Theorem 1) reveals that (5.2) also holds for those models with long range jumps or with non-finite range dependent rates.

Remark 2. We emphasize again that this theorem holds regardless of whether the family of measures ν satisfy the property detailed in Proposition 1.

A rather classical result immediately follows from Theorem 1, namely the quantity $\mathbf{E}_{\sigma^{\varrho, \lambda}} \omega_n(t)$ has uniform lower and upper bounds λ and ϱ , respectively, in the space $(n, t) \in \mathbb{Z} \times \mathbb{R}_0^+$. Also observe that for each fixed $t \in \mathbb{R}_0^+$, the function $n \mapsto \mathbf{E}_{\sigma^{\varrho, \lambda}} \omega_n(t)$ is *monotone non-increasing* in $n \in \mathbb{Z}$. The rest of the paper will basically discuss the far-reaching consequences of Theorem 1.

For the next theorem we take any function $\varphi : \mathcal{I} \rightarrow \mathbb{R}$ for which either condition $\varphi \geq 0$, or $\sum_{y \in \mathcal{I}} |\varphi(y)| < +\infty$ holds. Then we define $\Phi(x) = \sum_{y=\omega_{\min}}^x \varphi(y)$ and further assume $\mathbf{E}_{\sigma^{\varrho, \varrho}} \Phi(\omega_0(t)) < +\infty$.

Theorem 2 (Background as seen from the position of the second class particle). *Under the assumptions of Theorem 1 we have the following identity:*

$$\hat{\mathbf{E}} \varphi(\hat{\omega}_{Q(t)}(t)) = \frac{\mathbf{E}_{\sigma^{\varrho, \varrho}} \Phi(\omega_0(t)) - \mathbf{E}_{\sigma^{\lambda, \lambda}} \Phi(\omega_0(t))}{\varrho - \lambda}. \quad (5.3)$$

In plain words, this theorem tells that the law of $\hat{\omega}_{Q(t)}(t)$ for a $t \geq 0$, i.e. the particle occupation number at the position of the second class particle, can fully be captured by that of $\omega_0(t)$ of ω starting from $\sigma^{\varrho, \varrho}$ and then $\sigma^{\lambda, \lambda}$. In particular, if $\sigma^{\varrho, \varrho}$ and $\sigma^{\lambda, \lambda}$ are stationary distributions for the dynamics (2.2) then the background marginal one-site process $(\hat{\omega}_{Q(t)}(t), \hat{\eta}_{Q(t)}(t))_{t \geq 0}$, as seen from the position of the lone second class particle, is stationary. This can be thought of as another fact proving the intrinsicity of the marginal $\hat{\nu}^{\varrho, \lambda}$. Notice though that Theorem 2 does not say anything about the distribution of any site other than that of the second class particle, those are in general *not* stationary. A few very special cases of joint stationary distributions seen by the second class particle are described in [14, 5, 8] and references therein.

The conditions of Theorem 1 are mild enough to find many examples for the family ν fairly easily, though we highlight some choices below, also for demonstrating the strength of the assertion.

As the first rather simple example, we mention that all the deterministic marginals of the form $\nu^{\varrho}(x) = \mathbf{1}\{x = \varrho\}$ satisfy Assumption 1, where $\varrho \in \mathcal{D} := [\omega_{\min}, \omega_{\max}] \cap \mathbb{Z}$. For a more important family define the *Gibbs measures* as

$$\Gamma^{\theta}(x) := \frac{1}{Z(\theta)} \cdot \exp(\theta \cdot x + E(x)) \quad (x \in \mathcal{I}), \quad (5.4)$$

where $\theta \in \mathbb{R}$ is a generic real parameter, which is often referred to as the *chemical potential*; $E : \mathcal{I} \rightarrow \mathbb{R}$ is any function with appropriate asymptotic growth; finally, the *statistical- or partition sum* is $Z(\theta) = \sum_{y \in \mathcal{I}} \exp(\theta \cdot y + E(y))$.

It is known that the above defined Gibbs measures satisfy Assumption 1 (see [10, Appendix A] and also [11]). For the sake of completeness we restate this result below.

Proposition 3. *Assume that $\Gamma := (\Gamma^{\theta})_{\theta \in \mathcal{D}_c}$ forms a bunch of probability measures with finite variance, where \mathcal{D}_c is some open set of the reals. Then Γ satisfies all the conditions of Assumption 1. In particular, there is a bijection between the parameters $\theta \in \mathcal{D}_c$ and the densities $\varrho = \varrho(\theta) \in \mathbb{R}$; and for $\theta(\lambda) < \theta(\varrho)$, or equivalently for $\lambda < \varrho$, the measure $\Gamma^{\theta(\varrho)}$ stochastically dominates $\Gamma^{\theta(\lambda)}$.*

Due to the bijection claimed in the previous assertion we will change freely between the representations of the measure (5.4) either by the chemical potential $\theta = \theta(\varrho)$ or by the density $\varrho = \varrho(\theta)$.

In general notice that Γ^{ϱ} is *not* necessarily a stationary marginal of the dynamics (2.2). Following ideas of Coccozza-Thivent [13], for systems where Γ^{ϱ} is indeed stationary, a nice characterization theorem was established by M. Balázs et al., which we recall in the following.

Theorem 3 (M. Balázs et al.). *Let*

$$E(x) = \sum_{y=x+1}^0 \log(f(y)) - \sum_{z=1}^x \log(f(z)) \quad (x \in \mathcal{I}),$$

where $f : \mathbb{Z} \rightarrow \mathbb{R}^+$ is such that $f(x) = 1$ whenever $x \in \mathbb{Z} \setminus \mathcal{I}$. (The empty sum is as usual defined to be zero.) Suppose furthermore that:

- there are symmetric functions $s_p, s_q : \mathcal{I} \times \mathcal{I} \rightarrow \mathbb{R}_0^+$ such that for any $\omega \in \Omega$

$$r_i(\omega) = s_p(\omega_i, \omega_{i+1} + 1) \cdot f(\omega_i) \quad \text{and} \quad \ell_i(\omega) = s_q(\omega_i + 1, \omega_{i+1}) \cdot f(\omega_{i+1}), \quad (5.5)$$

where $s_p(\omega^{\min}, \cdot) \equiv 0$, $s_p(\cdot, \omega^{\max}) \equiv 0$, $s_q(\cdot, \omega^{\min}) \equiv 0$, and $s_q(\omega^{\max}, \cdot) \equiv 0$ hold whenever ω^{\min} or ω^{\max} is finite, otherwise they are non-zero except when r or ℓ is set to be zero (totally asymmetric case);

- for any $\omega \in \Omega$ and $i \in \mathbb{Z}$ for which $\omega_i = \omega_{i+3}$:

$$\begin{aligned} r_i(\omega) + r_{i+1}(\omega) + r_{i+2}(\omega) \\ + \ell_i(\omega) + \ell_{i+1}(\omega) + \ell_{i+2}(\omega) = r_i(\bar{\omega}) + r_{i+1}(\bar{\omega}) + r_{i+2}(\bar{\omega}) \\ + \ell_i(\bar{\omega}) + \ell_{i+1}(\bar{\omega}) + \ell_{i+2}(\bar{\omega}), \end{aligned} \quad (5.6)$$

where $\bar{\omega}_j = \omega_j$ for all $j \in \mathbb{Z}$ except that $\bar{\omega}_{i+1} = \omega_{i+2}$ and $\bar{\omega}_{i+2} = \omega_{i+1}$, that is the $(i+1)^{\text{th}}$ and $(i+2)^{\text{th}}$ coordinates of ω are reversed in $\bar{\omega}$.

Then the density parameterized product measure $\mathbf{\Gamma}^\varrho := \bigotimes_{i=-\infty}^{+\infty} \Gamma^\varrho$ is extremal among the translation-invariant stationary distributions of the attractive process with infinitesimal generator \mathcal{G} of (2.2).

Remark 3. Note that attractivity requires f to be monotone non-decreasing on $\mathcal{I} \setminus \{\omega^{\min}\}$.

Remark 4. The stationarity part of the previous assertion has been carried out thoroughly in [12], in which all the extremal translation-invariant stationary distributions were covered by examining the convergence region of the partition sum Z . For the ergodicity we will briefly comment on how Lemmas 7.2 and 7.3 of [10] established for the bricklayers' process can be modified to be handy for any process. First, it is not hard to see that Lemma 7.2 can be extended to the cases when (in any order) a positive and a negative second class particle start from next to each other. This results in that the probability of them colliding before any given time is (strictly) positive. Here the only required property of the underlying process is the continuity of its semigroup. Then in Lemma 7.3 ergodicity is carried out by showing that any invariant \mathcal{L}^2 function ψ w.r.t. $\mathbf{\Gamma}^{\theta(\varrho)}$ is constant. Now, by using (the extended version of) Lemma 7.2 it can be easily pointed out that adding $(+1, -1)$ (or $(-1, +1)$) to adjacent occupation numbers, whenever this change keeps the state space, does not modify the value of an invariant ψ . It follows that interchanging any two adjacent sites does not change the value of ψ under $\mathbf{\Gamma}^{\theta(\varrho)}$. The proof is then completed by the application of the Hewitt–Savage 0-1 law.

The conditions of Theorem 3 originate a wide range of (attractive) models among which we select some in Section 6. We emphasize again, neither (5.5) nor (5.6) is a requirement for Theorem 1 to hold.

Finally, in the above particular case (5.4), consider the measure $\hat{\nu}^{\varrho, \lambda}$ of (4.1) that is:

$$\begin{aligned} \hat{\nu}^{\varrho, \lambda}(x, y) &= \frac{1}{\varrho - \lambda} \sum_{z=\omega^{\min}}^y (\Gamma^{\theta(\lambda)}(z) - \Gamma^{\theta(\varrho)}(z)) \cdot \mathbf{1}\{x = y + 1\} \\ &= \frac{\theta(\varrho) - \theta(\lambda)}{\varrho - \lambda} \cdot \sum_{z=y+1}^{\omega^{\max}} \frac{\Gamma^{\theta(\varrho)}(z) - \Gamma^{\theta(\lambda)}(z)}{\theta(\varrho) - \theta(\lambda)} \cdot \mathbf{1}\{x = y + 1\} \end{aligned}$$

for $x, y \in \mathcal{I}$. Now, fixing ϱ and taking the limit as $\lambda \uparrow \varrho$ we obtain

$$(\hat{\nu}^\varrho)'(x, y) := \theta'(\varrho) \cdot \sum_{z=y+1}^{\omega^{\max}} (z - \varrho) \cdot \Gamma^{\theta(\varrho)}(z) \cdot \mathbf{1}\{x = y + 1\} \quad (x, y \in \mathcal{I}),$$

where it is easy to see that $\theta'(\varrho) = \frac{1}{\text{Var}(\omega_0)}$ for ω_0 distributed as $\Gamma^{\theta(\varrho)}$. (The empty sum is defined to be zero.) Observe that this probability measure $(\hat{\nu}^\varrho)'$ is just the marginal at the origin of the initial distribution that was used in [12, Theorem 2.2] to start a single second class particle from that position. Thus our treatment is in correspondence with results from [12]. As a side remark we mention without details that via a second order Taylor expansion as $\lambda \uparrow \varrho$ one can formally recover the covariance formula in [12, Theorem 2.2] directly from (5.2). Bounding the error terms that arise is straightforward when $|Z| < +\infty$, making this argument rigorous.

5.2 Hydrodynamics

This section is devoted to briefly recall some notions and results from hydrodynamics of (asymmetric) particle systems and with the help of these to draw further consequences of Theorem 1. For the sake of this section we suppose that the jump rates are given in the following special form:

$$r_i(\omega) = p(\omega_i, \omega_{i+1}) \quad \text{and} \quad \ell_i(\omega) = q(\omega_i, \omega_{i+1}),$$

where $p, q : \mathcal{I} \times \mathcal{I} \rightarrow \mathbb{R}_0^+$ are given deterministic functions. We assume that p (q) is monotone non-decreasing (non-increasing) in its first, and is monotone non-increasing (non-decreasing) in its second variable. In particular this condition guarantees for our processes to be attractive. Indeed, this is the point where the above particular choice for the rates is rather convenient for us. We note that many theorems below could also be adapted to more general setups, considering for e.g. long range jumps, where hydrodynamics as well as Theorem 1 are also available.

The idea behind the hydrodynamic limit has its roots in the law of large numbers, that is one may expect that the microscopic average density of interacting particles on some large scale behaves as a deterministic density field obeying the conservation law of the form:

$$\left. \begin{aligned} \partial_t u + \partial_x G(u) &= 0 \\ u(\cdot, 0) &= v(\cdot) \end{aligned} \right\} \quad (5.7)$$

where $u = u(x, t)$ is the (macroscopic) density with initial condition $v(\cdot)$ and G is the so-called *hydrodynamic flux*. This principle can be formulated in several ways which will be explicated in the following.

The *rescaled empirical measure* of a sequence of random configurations $(\omega^N)_{N \in \mathbb{N}}$ is defined as

$$\alpha^N(\omega^N, dx) = \frac{1}{N} \sum_{j \in \mathbb{Z}} \omega_j^N \mathbf{1}_{\{j/N \in dx\}} \quad (N \in \mathbb{N}).$$

Now, a deterministic bounded Borel measurable function v on \mathbb{R} is the *density profile* of $(\omega^N)_{N \in \mathbb{N}}$, if $\alpha^N(\omega^N, dx)$ converges to $v(x) dx$ as $N \rightarrow +\infty$ for all $x \in \mathbb{R}$, in probability as a random object, and in the topology of vague convergence as a measure, meaning that

$$\lim_{N \rightarrow +\infty} \mathbf{P}^N \left(\left| \frac{1}{N} \sum_{j \in \mathbb{Z}} \psi(j/N) \cdot \omega_j^N - \int_{x \in \mathbb{R}} \psi(x) \cdot v(x) dx \right| > \varepsilon \right) = 0$$

is required to hold for each $\varepsilon > 0$ and with all continuous test function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ of compact support.

Definition 2. A sequence of processes $(\omega^N(t))_{t \geq 0, N \in \mathbb{N}}$, generated by \mathcal{G} of (2.2) with random initial configurations $(\omega_0^N)_{N \in \mathbb{N}}$ exhibits a *hydrodynamic limit* u , if $(\omega^N(Nt))_{N \in \mathbb{N}}$ has density profile $u(\cdot, t)$ for every $t \geq 0$, where u is a (weak) solution to the problem (5.7).

We note that the hydrodynamic limit just defined is also referred to as the *weak conservation of local equilibrium* (cf. [24, Chapter 4]).

In the following, we will make the choice for ω_0^N to be of distribution $\sigma^{\varrho, \lambda}$ of (5.1) for all $N \in \mathbb{N}$, where Assumption 1 is satisfied. It then easily follows that $(\omega_0^N)_{N \in \mathbb{N}}$ has the *Riemannian density profile*

$$v(x) = \begin{cases} \varrho, & \text{if } x \leq 0, \\ \lambda, & \text{if } x > 0. \end{cases} \quad (5.8)$$

Under mild assumptions on G there exists a unique *entropy solution* u to the problem (5.7) with (5.8) as initial condition. It is also known that for each $t \geq 0$, this weak solution is continuous apart from a finite set of jump discontinuities (shocks), where we define $u(\cdot, t)$ to be left-continuous. For concepts and results in hyperbolic conservation laws, which were omitted here, we refer to [3] and further references therein (see also [23]).

In what follows some exact results on hydrodynamics will be collected concerning the above setting. The first general result is from [30].

Theorem 4 (F. Rezakhanlou). *Assume that the rate functions p, q are bounded and that all the conditions of Theorem 3 hold. Furthermore set the initial measure $\sigma^{\varrho, \lambda}$ to be of stationary marginals. Then $(\omega^N(t))_{t \geq 0, N \in \mathbb{N}}$ exhibits a hydrodynamic limit u , where u is the unique entropy solution to (5.7) with initial datum (5.8), and the hydrodynamic flux G is of the form*

$$G(\varrho) = \mathbf{E}_{\Gamma^{\varrho}} [p(\omega_0, \omega_1) - q(\omega_0, \omega_1)]. \quad (5.9)$$

In addition, the limit

$$\lim_{N \rightarrow +\infty} \mathbf{E}_{\sigma^{\varrho, \lambda}} \left[\frac{1}{N} \sum_{j \in \mathbb{Z}} \psi(j/N) \cdot \varphi(\tau_j \omega(Nt)) \right] = \int_{x \in \mathbb{R}} \psi(x) \cdot \mathbf{E}_{\Gamma^{u(x, t)}} \varphi(\omega) dx \quad (5.10)$$

also holds for every continuous ψ of compact support and any cylinder function $\varphi : \Omega \rightarrow \mathbb{R}$.

In the above result we are much restricted for the marginals of the initial measure to be chosen properly. However, this was far more generalized by C. Bahadoran et. al. in [3] for systems of bounded particle numbers per site.

Theorem 5 (C. Bahadoran, H. Guiol, K. Ravishankar and E. Saada). *Suppose that Assumption 1 holds and that both ω^{\min} and ω^{\max} are finite. Then $(\omega^N(t))_{t \geq 0}$ exhibits a hydrodynamic limit u with some Lipschitz continuous hydrodynamic flux G for every $\sigma^{\varrho, \lambda}$ of (5.1), where u is the unique entropy solution to (5.7) with initial datum (5.8).*

Remark 5. We refer to [3] for the detailed definition of G in the general case. Indeed, the previous assertion holds in even more general context as well as with sharper conclusions, for details consult [3] and [4].

Remark 6. Thanks to the step initial condition, by [3, Remark 2., pp. 1347], we can extend the previous theorem for those *unbounded systems* described in Theorem 3 where the rates are bounded. We understand from informal communications that these results can further be generalized to models with unbounded rates as well.

Our ultimate goal would be to conclude that the rescaled quantity $\mathbf{E}_{\sigma^{\varrho, \lambda} \omega_{[Nx]}(Nt)}$ also converges, where $[\cdot]$ denotes the integer part function. This, however, does not appear to be an immediate consequence of the above theorems. But C. Landim [25] has elaborated a set of assumptions under which this consequence eventually holds (note also [24, Proposition 0.6, Chapter 6]). We are going to recapitulate this result below to be formulated in our special context with sharper conclusions, outlining its proof in Section 5.

Proposition 4. *Suppose that the process with infinitesimal generator (2.2) exhibits a density parameterized, stochastically ordered and continuous family $(\pi^{\varrho})_{\varrho \in \mathcal{R}}$ of translation-invariant stationary distributions, where $\mathcal{R} \subset \mathbb{R}$ is such that $\varrho_{\min} < \varrho_{\max} \in \mathcal{R}$. Fix a cylinder function $\varphi : \Omega \rightarrow \mathbb{R}$, being either bounded or monotone non-decreasing, such that $\mathbf{E}_{\pi^{\varrho_{\max}}} |\varphi|(\omega) < +\infty$. Assume furthermore that the convergence*

$$\lim_{N \rightarrow +\infty} \mathbf{E}_{\tau_{[N\varepsilon]} \sigma^{\varrho, \lambda}} \left[\frac{1}{N} \sum_{j \in \mathbb{Z}} \psi(j/N) \cdot \varphi(\tau_j \omega^{\varepsilon, N}(Nt)) \right] = \int_{x \in \mathbb{R}} \psi(x) \cdot \mathbf{E}_{\pi^{u^{\varepsilon}(x, t)}} \varphi(\omega) dx \quad (5.11)$$

takes place for every $\varepsilon \in \mathbb{R}$ and continuous $\psi : \mathbb{R} \rightarrow \mathbb{R}$ of compact support with some uniformly bounded family of functions $(u^{\varepsilon})_{\varepsilon \in \mathbb{R}}$ for which $u^{\varepsilon} : \mathbb{R} \times \mathbb{R}_0^+ \rightarrow [\varrho_{\min}, \varrho_{\max}]$ is monotone non-increasing for each fixed $t \in \mathbb{R}_0^+$ and for every continuity point $x \in \mathbb{R}$ of $u^0(\cdot, t) : \lim_{\varepsilon \rightarrow 0} u^{\varepsilon}(x, t) = u^0(x, t)$. Then we have

$$\lim_{N \rightarrow +\infty} \mathbf{E}_{\sigma^{\varrho, \lambda}} \varphi(\tau_{[Nx]} \omega(Nt)) = \mathbf{E}_{\pi^{u^0(x, t)}} \varphi(\omega) \quad (5.12)$$

for every continuity point x of $u^0(\cdot, t)$.

By continuity of the set $(\pi^\varrho)_{\varrho \in \mathcal{R}}$ we mean that if $\varrho_n \rightarrow \varrho$ as $n \rightarrow +\infty$, where $\varrho_n, \varrho \in \mathcal{R}$, then $\pi^{\varrho_n} \rightarrow \pi^\varrho$ in the weak sense. Furthermore, the monotonicity of φ preserves the coordinate-wise order of configurations $\omega, \eta \in \Omega$, that is if $\omega \geq \eta$ then $\varphi(\omega) \geq \varphi(\eta)$. Notice that the convergence in (5.12) is also called as the *conservation of local equilibrium* (cf. [24, Chapter 1]). We can now state the main result of this section.

Theorem 6 (Speed of the second class particle). *Under the assumptions of Theorem 1 we have*

$$\lim_{N \rightarrow \infty} \hat{\mathbf{P}} \left\{ \frac{Q(Nt)}{N} \leq x \right\} = \frac{\varrho - u(x, t)}{\varrho - \lambda} \quad (5.13)$$

for every $x \in \mathbb{R}$ being a continuity point of $u(\cdot, t)$, provided that either conditions of Theorems 4 or 5 hold.

In particular we highlight that for systems with bounded occupation numbers the above limit distribution (5.13) holds for all choice of marginal distributions ν satisfying Assumption 1. By Remark 6 these can be extended to the unbounded models of Theorem 3.

In some cases concavity or convexity of the hydrodynamic flux G has been established, and it is then well understood that the Riemann initial condition (5.8) develops shock or rarefaction fan solutions, depending on the order of ϱ and λ , and on concavity or convexity of G . In a shock, the limiting probability (5.13) is of 0-1 form which means convergence of the scaled second class particle position to the deterministic velocity of the shock. In a rarefaction fan we have convergence to a random velocity. This randomness is uniform for the asymmetric simple exclusion process, as u is a linear function of its first – spatial – argument. This has been observed by P. A. Ferrari and C. Kipnis [17], but the distribution can be of different form for other processes. See also P. Gonçalves [21] for a recent extension via coupling of [17] to constant rate zero-range.

We emphasize that there are attractive models with product-form stationary distributions but with non-concave, non-convex fluxes. In the associated conservation laws coexistence of shocks and rarefaction fans is possible, in which cases our result shows that the limit distribution of the velocity of the second class particle is mixed with a discrete mass and a continuous counterpart (see Section 6).

5.3 Symmetric processes

Since Theorem 1 does not pose any specific condition on the rates we can naturally consider symmetric processes as well. To keep notation simple we will adopt the setup of the previous Subsection 5.2, that is $p, q : \mathcal{I}^2 \rightarrow \mathbb{R}_0^+$ denote deterministic functions for which $r_i(\omega) = p(\omega_i, \omega_{i+1})$ is the right rate while $\ell_i(\omega) = q(\omega_i, \omega_{i+1})$ is the left rate of the process ($\omega \in \Omega$). In our context, being symmetric means that the relation $q(\omega_i, \omega_{i+1}) = p(\omega_{i+1}, \omega_i)$ is also satisfied between p and q for each $\omega \in \Omega$. Note that the attractivity condition is still up, that is p is required to be monotone non-increasing (non-decreasing) in its first (second) variable.

We say that a symmetric attractive process is *gradient* if there exists a *cylinder* function $g : \Omega \rightarrow \mathbb{R}$ for which

$$p(\omega_i, \omega_{i+1}) - p(\omega_{i+1}, \omega_i) = g(\tau_i \omega) - g(\tau_{i+1} \omega) \quad (5.14)$$

holds for every $i \in \mathbb{Z}$ and $\omega \in \Omega$. Usually it is more convenient and simpler to deal with (attractive) gradient systems. For such systems the key quantity turns out to be the *diffusivity coefficient* d which is defined to be $d(\varrho) = \mathbf{E}_{\pi^\varrho} g(\omega)$, where π^ϱ is some stationary distribution of the process with density ϱ . Note the difference between d and the hydrodynamic flux G (defined in (5.9)) being its hyperbolic counterpart.

The concepts of hydrodynamics of Subsection 5.2 can be repeated here, except that this time the relevant *time scale* is N^2 instead of N . Hence the macroscopic behavior of the density field is described by a *parabolic* partial differential equation of the form

$$\left. \begin{aligned} \partial_t u &= \frac{1}{2} \Delta d(u) \\ u(\cdot, 0) &= v(\cdot) \end{aligned} \right\} \quad (5.15)$$

where v is defined in (5.8). In general note that it is not so obvious for (5.15) to have a unique bounded (classical or weak) solution due to the discontinuity of v and the smoothness of d .

However we will skip investigating this issue by assuming that there always exists a unique weak solution to (5.15). Now, we can formulate the main result here.

Theorem 7 (Limit distribution of second class particle in diffusive environment). *Suppose all the conditions of Theorem 1 and that the sequence of gradient processes $(\omega^N(t))_{t \geq 0, N \in \mathbb{N}}$, starting from (5.1), exhibits a hydrodynamic limit u , which is the unique weak solution to (5.15). Now, for $\omega^{\min} > -\infty$, we have*

$$\lim_{N \rightarrow \infty} \hat{\mathbf{P}} \left\{ \frac{Q(Nt)}{\sqrt{N}} \leq x \right\} = \frac{\varrho - u(x, t)}{\varrho - \lambda} \quad (5.16)$$

to hold for every continuity point $x \in \mathbb{R}$ of $u(\cdot, t)$, provided that $\mathbf{E}_{\sigma^{e, e}} \omega_0(t)^2 < +\infty$ for $t \geq 0$.

Remark 7. The hydrodynamic limit of gradient systems is widely known (see [24], [34, Chapter 8] and many references therein, particularly [20] and [26]) which methods partially extend to non-gradient systems [29] as well. We will highlight some examples in Section 6.

5.4 Collision probability

This short section focuses on the interaction of two second class particles of opposite charges dropped into the system initially. Denote by $\mathcal{N}(t)$ the total number of second class particles present in the system at time t . For the long-time behavior of \mathcal{N} we will state one result below.

Theorem 8 (Collision probability of second class particles). *Assume that ω^{\min} and ω^{\max} are finite numbers. Let $(\hat{\omega}, \hat{\eta})$ be a pair of attractive systems starting from the deterministic initial configurations*

$$\hat{\omega}_0 = \hat{\eta}_0 - \delta_0 + \delta_1, \quad \hat{\eta}_0 = \omega^{\max} \mathbf{1}\{i \leq 0\} + \omega^{\min} \mathbf{1}\{i > 0\},$$

and evolving according to the basic coupling. Then

$$\hat{\mathbf{P}}\{\mathcal{N}(t) = 2 \text{ for all } t \geq 0\} \geq \frac{1}{r_0(\hat{\eta})} \cdot \bar{G}(1) =: C_0, \quad (5.17)$$

where $\bar{G}(1) = \limsup_{N \rightarrow +\infty} \mathbf{E}_{\hat{\eta}_0}[r_0(\hat{\eta}(N)) - \ell_0(\hat{\eta}(N))]$, while $\hat{\mathbf{P}}$ denotes the associated probability of $(\hat{\omega}(t), \hat{\eta}(t))_{t \geq 0}$.

In particular, if the dynamics is totally asymmetric, that is $\ell_0 \equiv 0$, then $C_0 > 0$ holds. On the other hand, considering one of models described by Theorem 3 we have $\bar{G}(1) = G(u(0, 1))$ provided that 0 is a continuity point of $u(\cdot, 1)$, where G is defined as in (5.9) and u is the unique entropy solution to (5.7) with initial datum $v(x) = \omega^{\max} \mathbf{1}\{x \leq 0\} + \omega^{\min} \mathbf{1}\{x > 0\}$ ($x \in \mathbb{R}$).

In plain words the above theorem tells that two second class particles of distinct charges initially placed at lattice points 0 and 1 will never meet with positive probability, provided that the constant C_0 of (5.17) is positive. A careful look at the proof reveals that the event on the left hand-side can in fact be replaced by the smaller one {the two second class particles are on the two sides of the origin at time t }.

For the asymmetric simple exclusion process with $\mathcal{I} = \{0, 1\}$ and rate functions $r_i(\omega) = \bar{p} \cdot \omega_i \cdot (1 - \omega_{i+1})$ and $\ell_i(\omega) = (1 - \bar{p}) \cdot \omega_{i+1} \cdot (1 - \omega_i)$ ($i \in \mathbb{Z}$), where $\bar{p} \in (\frac{1}{2}, 1]$, we recover the result [17, Theorem 2], if $\bar{p} = 1$. Indeed, we know exactly from [16, Theorem 2.3] that for each $\bar{p} \in (\frac{1}{2}, 1]$

$$\hat{\mathbf{P}}\{\mathcal{N}(t) = 2 \text{ for all } t \geq 0\} = \frac{2\bar{p} - 1}{3\bar{p}}$$

for which (5.17), C_0 being $\frac{2\bar{p}-1}{4\bar{p}}$, gives a non-sharp lower bound.

6 Particular examples

We have selected some particular models in order to demonstrate the versatility of the results of Section 5. Our general framework contains several well studied examples like the (totally) asymmetric simple exclusion process or the class of zero-range processes. Below, we first list some well known asymmetric and then symmetric processes with some additional description. In the following we fix two reals \bar{p}, \bar{q} for which $0 \leq \bar{p} < \bar{q} \leq 1$ and $\bar{p} + \bar{q} = 1$ hold.

Generalized exclusion processes. Many systems with bounded occupations lie in this class but we only illustrate two of them.

The first one is the 2-type model of [9]. Briefly, it is a totally asymmetric process with $\mathcal{I} = \{-1, 0, +1\}$ and rates

$$p(0, 0) = c, \quad p(0, -1) = p(+1, 0) = \frac{1}{2}, \quad p(+1, -1) = 1 \quad (6.1)$$

and $q \equiv 0$ using the notation of Subsection 5.2. In plain words the dynamics consists of the following simple rules: two adjacent holes can produce an antiparticle-particle pair (*creation*), (anti)particles can hop to the (negative) positive direction (*exclusion*), and when a particle meets an antiparticle they can annihilate each other (*annihilation*). It is easy to see that the process is attractive only if $c \leq \frac{1}{2}$. It then lies in the range of Theorem 3 having product-form ergodic measures.

The hydrodynamic behavior, but *not* the second class particles, of the model has been thoroughly investigated by M. Balázs, A. L. Nagy, B. Tóth and I. Tóth [9]. In that article the hydrodynamic flux G was explicitly calculated, which turned out to be neither concave nor convex in some region of the parameter space. Hence the entropy solution of the hydrodynamic equation can produce various mixtures of rarefaction fans and shock waves. By (5.13) it implies that the limit distribution of the second class particle can have both continuous and discrete parts which will be demonstrated in the following. Using the results of [9] we can basically evaluate (5.13) of Theorem 6 in each case but we highlight only two of them below. For sake of simplicity we let $\varrho = 1$ and $\lambda = -1$.

Concave flux. In the region $\frac{1}{16} \leq c \leq \frac{1}{2}$: the hydrodynamic flux G is concave [9]. In particular for $c = \frac{1}{16}$, Figure 1 demonstrates how the one parameter family of limit distributions of the second class particle evolves in time as $t \in [0, 1]$. We notice that for all $t > 0$ the cumulative distribution function F_t^Q is *continuous* but has a vertical “slope” at the origin. Thus its density is unbounded around zero.

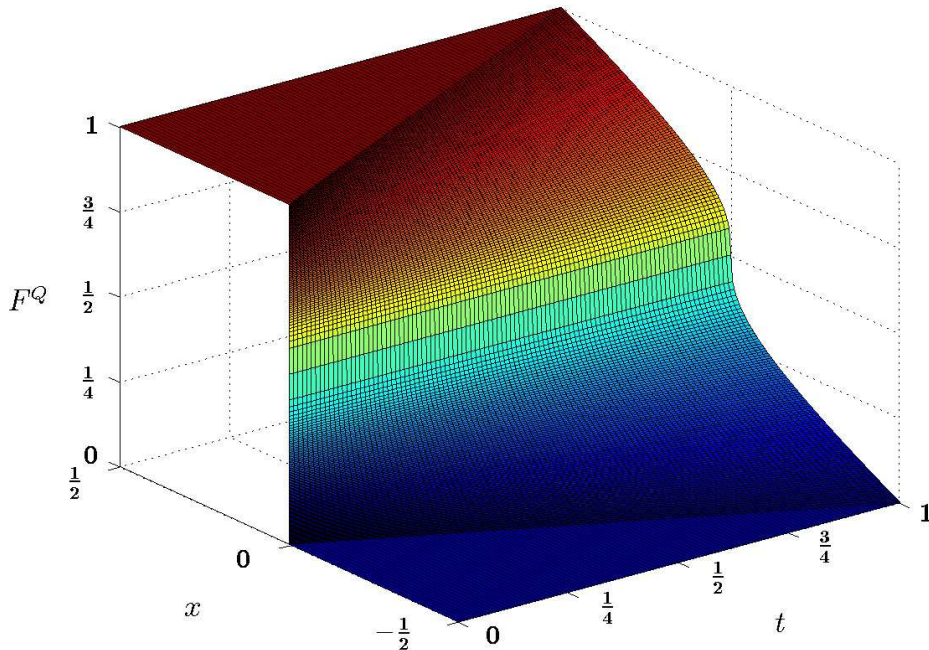


Figure 1. The limit distribution of second class particle when $c = \frac{1}{16}$, the hydrodynamic flux G being (non-strictly) concave. In particular, a vertical slice of the surface gives a limit distribution $F_t^Q(\cdot) := \lim_{N \rightarrow +\infty} \mathbf{P}\{\frac{1}{N}Q(Nt) \leq \cdot\}$ for a fixed t .

Non-convex flux. In the region $0 < c < \frac{1}{16}$: the hydrodynamic flux G is neither concave nor convex [9]. As a particular example, for $c = \frac{1}{324}$ the model can develop a (non-linear) *rarefaction fan – shock – rarefaction fan profile* in the hydrodynamic limit. The second class particle then may stick into the shock with probability $v_{\max} := \frac{1}{2}\sqrt{\frac{1-16c}{1-4c}}$ or it follows a continuously chosen characteristics in one of the regions of the rarefaction fan. Figure 2 demonstrates this behavior as $t \in [0, 1]$.

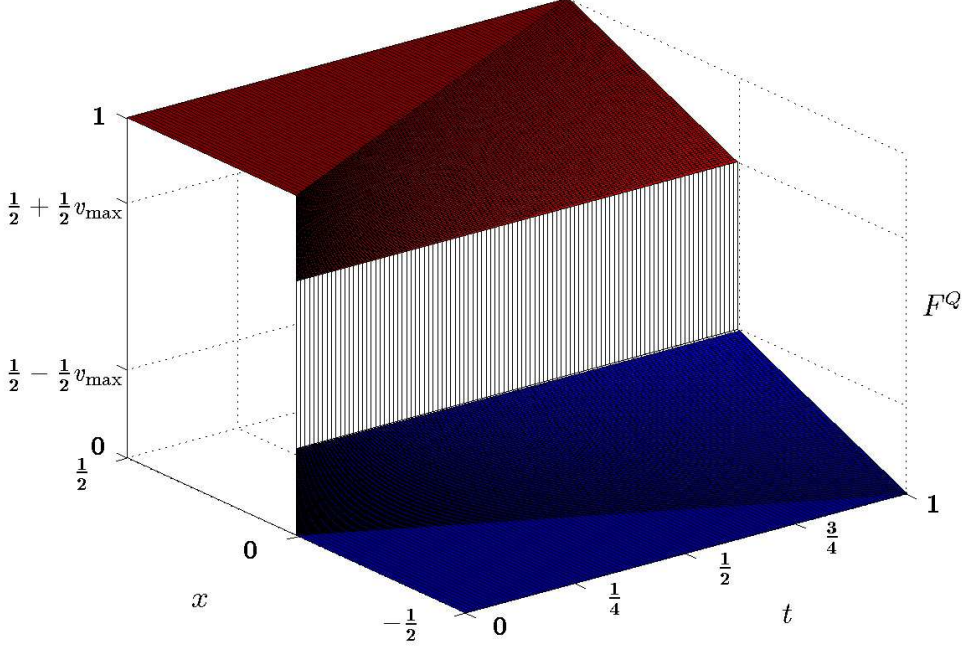


Figure 2. The limit distribution of the second class particle in the case of $c = \frac{1}{324}$ when the hydrodynamic flux G is neither concave nor convex. A vertical slice of the surface gives a particular limit distribution.

Relying again on [9] we finally note that one can explicitly calculate the estimate of Theorem 8: the collision of two second class particles starting from the rarefaction fan has at least probability $C_0 = G(0) = \frac{\sqrt{c}}{1+2\sqrt{c}} \in (0, \frac{1}{2+\sqrt{2}}]$ in this model.

Another example we mention is the K -exclusion process, where K is any positive integer. Set $\mathcal{I} = \{0, 1, \dots, K-1, K\}$ and let the rates be

$$r_i(\omega) = \bar{p} \cdot \mathbf{1}\{0 < \omega_i\} \cdot \mathbf{1}\{\omega_{i+1} < K\}, \quad \ell_i(\omega) = \bar{q} \cdot \mathbf{1}\{0 < \omega_{i+1}\} \cdot \mathbf{1}\{\omega_i < K\}.$$

In particular for $K = 1$ we obtain the asymmetric simple exclusion process with the family of Bernoulli product measures as extremal translation-invariant stationary distributions which work well for Assumption 1 and thus recover the result of [17]. For $K > 1$ much less is known (the assumptions of Theorem 3 cease to hold). In particular, it is not known whether its density parameterized translation-invariant extremal stationary distributions span the range $[0, K]$. They are proved to exist for some closed parameter set $\mathcal{R} \subset [0, K]$ (see [3, Corollary 2.1]). The structure of these measures is also unknown. The model, however, exhibits a hydrodynamic limit resulting in a conservation law with a concave flux G (see Theorem 5 and also [33]). For G only some qualitative properties have been established (see [33]). Nevertheless, one can still apply Theorems 1 and 6 with some product initial distributions that satisfy Assumption 1.

Zero range processes. Let $\omega^{\min} = 0$ and $\omega^{\max} = +\infty$. The jump rates are defined as

$$r_i(\omega) = \bar{p} \cdot f(\omega_i), \quad \ell_i(\omega) = \bar{q} \cdot f(\omega_{i+1}),$$

where $f : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$ is a monotone non-decreasing function with at most linear growth and with $f(0) = 0$. This family satisfies all the assumptions of Theorem 3, hence hydrodynamics follows via Theorem 4. For simplicity we spell out two totally asymmetric examples ($\bar{p} = 1$). The hydrodynamic flux is then given by $G(\varrho) = \exp(\theta(\varrho))$. The two most well-known special cases we consider are the ones of constant and linear rates: $f(x) = \mathbb{1}\{x > 0\}$ and $f(x) = x$, respectively. In the former case the extremal translation-invariant stationary distributions are of product form with geometric site-marginals while in the latter the Poisson distribution takes over this role. $G(\varrho)$ is $\varrho \cdot (1 + \varrho)^{-1}$ and ϱ , respectively.

A straightforward computation then shows (see [23, Section 2.2, pp. 30–36]), that for the totally asymmetric constant rate zero-range process, (5.13) takes the form:

$$\lim_{N \rightarrow +\infty} \mathbf{P} \left\{ \frac{Q(Nt)}{N} \leq x \right\} = \begin{cases} 0 & \text{if } (1 + \varrho)^2 \leq \frac{t}{x}; \\ \frac{\varrho + 1}{\varrho - \lambda} - \frac{1}{\varrho - \lambda} \sqrt{\frac{t}{x}} & \text{if } (1 + \lambda)^2 \leq \frac{t}{x} < (1 + \varrho)^2; \\ 1 & \text{if } (1 + \lambda)^2 > \frac{t}{x}. \end{cases}$$

Due to Remark 6 we in fact get this law for the limit velocity of the second class particle no matter how we choose, still under Assumption 1, the initial marginals of (5.1).

The totally asymmetric linear rate zero-range process is a much easier story (independent walkers). In that case Q is a unit rate Poisson process in agreement with the unique entropy solution of the transport equation $\partial_t u + \partial_x u = 0$, being $u(x, t) = u_0(x - t)$. No novelty here, of course.

Deposition models. Now, let $\omega^{\min} = -\infty$ and $\omega^{\max} = +\infty$. The generalized bricklayers' process is defined to have rates:

$$r_i(\omega) = \bar{p} \cdot (f(\omega_i) + f(-\omega_{i+1})), \quad \ell_i(\omega) = \bar{q} \cdot (f(-\omega_i) + f(\omega_{i+1})),$$

where $f : \mathbb{Z} \rightarrow \mathbb{R}^+$ is any monotone non-decreasing function, also having the property that $f(x) \cdot f(1 - x) = 1$ for all $x \in \mathbb{Z}$. This family was first introduced and investigated in [5]. For rates growing at most exponentially, existence of dynamics was showed in the totally asymmetric case [10]. However, results concerning hydrodynamics have not been established yet for unbounded rates, and so Theorem 6 is conditional in this case.

In particular, we obtain the *totally asymmetric exponential bricklayers' process* if we set $\bar{p} = 1$ and $f(x) = \exp(\beta(x - 1/2))$ ($x \in \mathbb{Z}$), where β is a fixed positive real. Then Theorem 3 shows that the product of *discrete Gaussian distributions*, defined as

$$\Gamma^{\theta(\varrho)}(x) = \frac{1}{Z(\theta(\varrho))} \cdot \exp\left(-\beta \cdot (x - \theta(\varrho)/\beta)^2/2\right) \quad (x \in \mathbb{Z}),$$

where $\varrho \in \mathbb{R}$, $\theta \in \mathbb{R}$ and $Z(\theta(\varrho)) = \sum_{y \in \mathbb{Z}} \exp\left(-\beta \cdot (y - \theta(\varrho)/\beta)^2/2\right)$, is stationary. This measure enjoys the remarkable property that

$$\Gamma^{\theta(\varrho)}(x) = \Gamma^{\theta(\varrho) - \beta}(x - 1) = \Gamma^{\theta(\varrho - 1)}(x - 1) \quad (\varrho \in \mathbb{R}, x \in \mathbb{Z}).$$

This fact indeed implies that the discrete Gaussian satisfies even condition (4.4) of Proposition 1. Articles [5, 7, 8] made also good use of the previous identity for exploring special random walking shock-like product distributions that also include the second class particle. Finally, if we use $\Gamma^{\theta(\varrho)}$ and $\Gamma^{\theta(\lambda)}$ as initial marginals with $\varrho = \lambda + 1 \in \mathbb{R}$, then the measure $\hat{\nu}^{\varrho, \lambda}$ gets a particularly simple form, namely $\hat{\nu}^{\varrho, \lambda}(x, y) = \Gamma^{\theta(\lambda)}(y) \cdot \mathbb{1}\{x = y + 1\}$.

Symmetric processes. One of the most studied gradient processes is the *simple symmetric exclusion process* with $\mathcal{I} = \{0, 1\}$ and with rates

$$r_i(\omega) = \mathbb{1}\{\omega_i = 1, \omega_{i+1} = 0\}, \quad \ell_i(\omega) = \mathbb{1}\{\omega_{i+1} = 1, \omega_i = 0\}.$$

In this case it is straightforward that the motion of the second class particle is a simple symmetric random walk. Hence its scaling results in the normal distribution. Our machinery implies this very simple fact and is clearly an overshoot for this case.

Nevertheless, the scaling limit of the second class particle in our scenario becomes far less trivial for more sophisticated symmetric models like the next one. Let $\mathcal{I} = \{-1, 0, +1\}$ and define the rates as

$$\begin{aligned} r_i(\omega) &= (|\omega_i| + |\omega_{i+1}|) \cdot \mathbb{1}\{-1 < \omega_i\} \cdot \mathbb{1}\{\omega_{i+1} < 1\} + c \cdot \mathbb{1}\{\omega_i = \omega_{i+1} = 0\}; \\ \ell_i(\omega) &= (|\omega_{i+1}| + |\omega_i|) \cdot \mathbb{1}\{-1 < \omega_{i+1}\} \cdot \mathbb{1}\{\omega_i < 1\} + c \cdot \mathbb{1}\{\omega_i = \omega_{i+1} = 0\}, \end{aligned}$$

with $0 < c \leq 1$. Note the essential difference between this and K -exclusion from before. We remark that this model is the symmetrized version of (6.1).

The above defined processes both enjoy product-form stationary distributions by Theorem 3. Their gradient function is $g(\omega) = \omega_0$ that is the macroscopic behavior is described by the homogeneous heat equation $\partial_t u = \frac{1}{2} \Delta u$ in both cases. Hence

$$\lim_{N \rightarrow +\infty} \mathbf{P} \left\{ \frac{Q(Nt)}{\sqrt{N}} \leq x \right\} = \Phi \left(\frac{x}{\sqrt{t}} \right),$$

where Φ denotes the cumulative distribution function of a standard normal variable.

Symmetric zero-range processes. Let $\omega^{\min} = 0$ and $\omega^{\max} = +\infty$. Then define the rates as

$$r_i(\omega) = f(\omega_i), \quad \ell_i(\omega) = f(\omega_{i+1})$$

with a common, monotone non-decreasing function $f : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$ such that $f(0) = 0$. In this case the gradient condition (see (5.14)) works with $g = f$. And so the macroscopic equation reads as $\partial_t u = \frac{1}{2} \Delta d(u)$, where $d(\varrho) = \mathbf{E}_{\Gamma^e} f(\omega)$ and Γ^e can be read off from Theorem 3. We highlight the constant and the linear rate case, that is when $f(x) = \mathbb{1}\{x > 0\}$ and $f(x) = x$, respectively. In the latter case we get normal behavior for Q , while in the former case

$$\partial_t u = \frac{1}{2} \partial_x \left(\frac{1}{(1+u)^2} \partial_x u \right) \quad (6.2)$$

has to be solved. By change of variables one can deduce and then easily verify that the unique

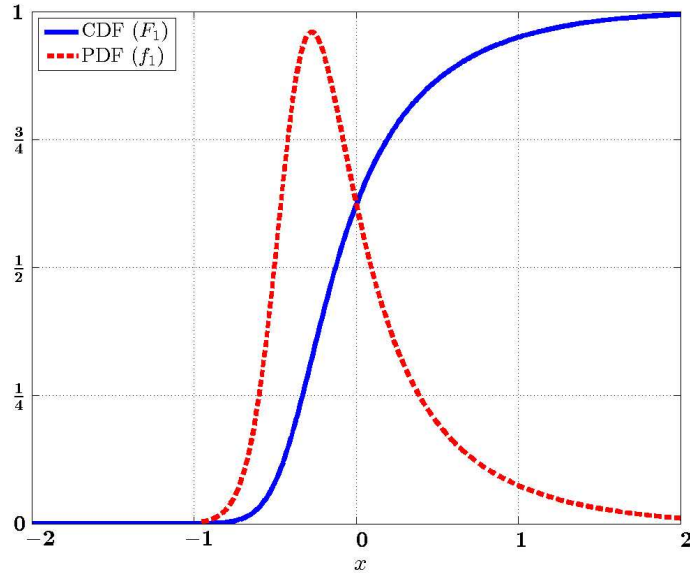


Figure 3. The limit distribution of second class particle at $t = 1$ in constant rate symmetric zero-range process with $\lambda = 0$ and $\varrho = 3$.

bounded classical solution of (6.2) with initial condition (5.8) is $u(x, t) = (h^{-1})'(x/\sqrt{t}) - 1$, where

$$h(y) = \frac{y}{1+\varrho} + \left(\frac{1}{1+\lambda} - \frac{1}{1+\varrho} \right) \cdot (\varphi(y) + y \cdot \Phi(y)) \quad (y \in \mathbb{R}),$$

φ and Φ denote the probability density function and the cumulative distribution function of a standard normal variable, while h^{-1} is the inverse function of h . This gives that the limit distribution function of (5.16) is of the form:

$$F_t(x) := \lim_{N \rightarrow \infty} \hat{\mathbf{P}} \left\{ \frac{Q(Nt)}{\sqrt{N}} \leq x \right\} = \frac{\varrho + 1}{\varrho - \lambda} - \frac{1}{\varrho - \lambda} \cdot \frac{1}{h'(h^{-1}(x/\sqrt{t}))},$$

thus its density function is

$$f_t(x) := F'_t(x) = \frac{1}{\sqrt{t}} \cdot \frac{1}{(1 + \varrho) \cdot (1 + \lambda)} \cdot \frac{\varphi(h^{-1}(x/\sqrt{t}))}{(h'(h^{-1}(x/\sqrt{t})))^3}.$$

Observe that the limit distribution has always zero expectation though the position of the second class particle is *not* a martingale. Furthermore, F_t is positively skewed and that it has negative mode as well as median implying that the second class particle is more likely to have a *negative speed*. In other words, it locally sees more first class particle before him rather than behind him which seems an unexpected phenomenon. An illustration can be found on Figure 3.

7 Proofs

We start by proving the results of Section 4.

Proof of Proposition 1. Assume that $\varrho - 1 \leq \lambda < \varrho$ are given, where $\varrho, \lambda \in \mathcal{D}$. To be able to construct a joint probability measure $\nu^{\varrho, \lambda}$ with the corresponding marginal distributions ν^{ϱ} and ν^{λ} in such a way that $\nu^{\varrho, \lambda}(\{(x, y) : x - y \in \{0, 1\}\}) = 1$ also holds we must certainly have that $\nu^{\varrho, \lambda}(x, y) = 0$ whenever $x - y \notin \{0, 1\}$. For sake of simplicity the only non-vanishing probabilities are denoted by

$$c_{y,y} := \nu^{\varrho, \lambda}(\{(y, y)\}) \quad \text{and} \quad c_{y+1,y} := \nu^{\varrho, \lambda}(\{(y+1, y)\}),$$

where $y \in \mathcal{I}$ ($c_{\omega^{\max}+1, \omega^{\max}}$ is defined to be zero).

For the marginals to match we have the following constraints:

$$c_{y,y} + c_{y,y-1} = \nu^{\varrho}(\{y\}) \quad \text{and} \quad c_{y,y} + c_{y+1,y} = \nu^{\lambda}(\{y\}),$$

which implies that $c_{y+1,y} = c_{y,y-1} + \nu^{\lambda}(\{y\}) - \nu^{\varrho}(\{y\})$. It then follows by recursion that

$$c_{y+1,y} = \nu^{\lambda}(\{z : z \leq y\}) - \nu^{\varrho}(\{z : z \leq y\}), \quad (7.1)$$

hence we obtain that

$$c_{y,y} = \nu^{\varrho}(\{z : z \leq y\}) - \nu^{\lambda}(\{z : z \leq y-1\}) \quad (7.2)$$

holds for every $y \in \mathcal{I}$. Since the last two expressions must be non-negative, these provide the stochastic dominance in Assumption 1 and the condition (4.4), respectively.

Finally, the identity $\sum_{y \in \mathcal{I}} c_{y+1,y} = \varrho - \lambda$ proves (4.5). ■

Proof of Proposition 2. First, in the geometric case, let $0 < p < p' \leq 1$,

$$\nu^{\varrho}(\{n\}) := p \cdot (1 - p)^n \quad \text{and} \quad \nu^{\lambda}(\{n\}) := p' \cdot (1 - p')^n \quad (0 \leq n \in \mathbb{Z})$$

be the weights of the Geometric distributions with mean $\varrho := 1/p - 1$ and $\lambda := 1/p' - 1$, respectively. It is an obvious fact that ν^{ϱ} stochastically dominates ν^{λ} , that is Assumption 1 holds. But condition (4.4) is violated, since the inequality

$$\begin{aligned} 0 &\leq \nu^{\varrho}(\{n : n \leq y\}) - \nu^{\lambda}(\{n : n \leq y-1\}) \\ &= \sum_{n=0}^y p \cdot (1 - p)^n - \sum_{n=0}^{y-1} p' \cdot (1 - p')^n = (1 - p')^y \left[1 - (1 - p') \left(\frac{1 - p}{1 - p'} \right)^{y+1} \right] \end{aligned}$$

cannot hold for all $0 \leq y \in \mathbb{Z}$ simultaneously.

In the Poisson case, let $0 \leq \lambda < \varrho < +\infty$,

$$\nu^\varrho(\{n\}) := \exp(-\varrho) \cdot \frac{\varrho^n}{n!} \quad \text{and} \quad \nu^\lambda(\{n\}) := \exp(-\lambda) \cdot \frac{\lambda^n}{n!} \quad (0 \leq n \in \mathbb{Z})$$

be the weights of the corresponding Poisson distributions. Their cumulative distribution functions are:

$$\nu^\varrho(\{n : n \leq y\}) = \frac{1}{y!} \int_{\varrho}^{+\infty} s^y \exp(-s) ds, \quad \nu^\lambda(\{n : n \leq y\}) = \frac{1}{y!} \int_{\lambda}^{+\infty} s^y \exp(-s) ds,$$

where $0 \leq y \in \mathbb{Z}$. These formulæ can be proven via induction on y using integration by parts. The above write-up shows the validity of Assumption 1 as well. Condition (4.4), however, does not hold, since the following inequality cannot hold for all $0 \leq y \in \mathbb{Z}$:

$$\begin{aligned} 0 &\leq \nu^\varrho(\{n : n \leq y\}) - \nu^\lambda(\{n : n \leq y-1\}) \\ &= \frac{1}{y!} \left[\int_{\varrho}^{+\infty} s^y \exp(-s) ds - \int_{\lambda}^{+\infty} y \cdot s^{y-1} \exp(-s) ds \right] \\ &= \exp(-\lambda) \cdot \frac{\lambda^y}{y!} \left[1 - \int_{\lambda}^{\varrho} \left(\frac{s}{\lambda} \right)^y \exp(\lambda - s) ds \right], \end{aligned}$$

where at the last inequality we took advantage of the integration by parts formula. ■

Next, we turn to the proofs of the main results of Section 5. First, we define the notion of *particle current* that will need subsequently.

Denote by \mathbb{S} the set of half integers, that is $\mathbb{S} := \mathbb{Z} + \frac{1}{2} = \{i + \frac{1}{2} : i \in \mathbb{Z}\}$. For each $\omega \in \Omega$ we assign a *height configuration*

$$\mathbf{h} = (\dots, h_{-\frac{3}{2}}, h_{-\frac{1}{2}}, h_{\frac{1}{2}}, h_{\frac{3}{2}}, \dots) \in \mathbb{Z}^{\mathbb{S}}$$

such that $\omega_i = h_k - k_{k+1}$ holds for $k = i - \frac{1}{2} \in \mathbb{S}$. Thus the negative discrete gradient of \mathbf{h} provides the system ω . Reversing this gives the heights as a function of ω :

$$h_k = \begin{cases} h_{\frac{1}{2}} - \sum_{i=1}^{k-\frac{1}{2}} \omega_i & \text{if } \frac{1}{2} < k \in \mathbb{S}; \\ h_{\frac{1}{2}} + \sum_{i=k+\frac{1}{2}}^0 \omega_i & \text{if } \frac{1}{2} > k \in \mathbb{S}, \end{cases} \quad (7.3)$$

except for a constant simultaneous shift in all of the height values. We fix this integration constant by postulating $h_{\frac{1}{2}}(0)$ to be zero initially. Then, the dynamics (2.1) translates to

$$\mathbf{h} \xrightarrow{r_i(\omega)} \mathbf{h} + \delta_{i+\frac{1}{2}}, \quad \mathbf{h} \xrightarrow{\ell_i(\omega)} \mathbf{h} - \delta_{i+\frac{1}{2}}.$$

Equivalently, $h_{\frac{1}{2}}(t) - h_{\frac{1}{2}}(0)$ denotes the number of *signed* particle hops that occurred above the bond $[0, 1]$ until time $t > 0$. In a similar fashion, one can think of $h_k(t)$ as the *signed particle current* through the space-time line $(\frac{1}{2}, 0) \rightarrow (k, t)$, where $k \in \mathbb{S}$ and $t \in \mathbb{R}^+$.

Proof of Theorem 1. Fix $t \geq 0$, $n \in \mathbb{Z}$ and let $k = n + \frac{1}{2}$. We start two processes ω and η with respective initial distributions $\sigma^{\varrho, \lambda}$ and $\tau_1 \sigma^{\varrho, \lambda}$, where recall (5.1) and that

$$\tau_1 \sigma^{\varrho, \lambda} = \bigotimes_{i=-\infty}^{-1} \nu^\varrho \otimes \bigotimes_{i=0}^{+\infty} \nu^\lambda, \quad (7.4)$$

which is just the left-shifted version of $\sigma^{\varrho, \lambda}$. The corresponding height functions are denoted by \mathbf{h} and \mathbf{g} .

Closely following [17], we will compute the following quantity in two different ways:

$$\mathbf{E}_{\sigma^e, \lambda} h_k(t) - \mathbf{E}_{\tau_1 \sigma^e, \lambda} g_k(t). \quad (7.5)$$

Define the *possibly signed* site-marginal

$$\bar{\nu}^{e, \lambda}(x, y) = \begin{cases} \nu^e(\{z : z \leq y\}) - \nu^\lambda(\{z : z \leq y - 1\}) & \text{if } x = y; \\ \nu^\lambda(\{z : z \leq y\}) - \nu^e(\{z : z \leq y\}) & \text{if } x = y + 1; \\ 0 & \text{otherwise,} \end{cases} \quad (x, y \in \mathcal{I}) \quad (7.6)$$

and the *possibly signed* product measure $\bar{\mu} := \bigotimes_{i=-\infty}^{-1} \nu^{e, e} \otimes \bar{\nu}^{e, \lambda} \otimes \bigotimes_{i=1}^{+\infty} \nu^{\lambda, \lambda}$, with $\nu^{e, e}$, $\nu^{\lambda, \lambda}$ defined in (4.2). Note that the marginal $\bar{\nu}^{e, \lambda}$ is the one we have just elaborated in (7.1)–(7.2). Proposition 1 tells that a coupling is possible with zero or one discrepancy if and only if $\bar{\nu}^{e, \lambda}$ is a probability measure (see (4.4)). This does not need to be the case, $\bar{\nu}^{e, \lambda}$, hence $\bar{\mu}$, might put negative mass for some initial configuration pairs. But thanks to Assumption 1, it may only put negative mass on coinciding initial configurations, that is when $\omega(0) \equiv \eta(0)$. In other words, $\bar{\nu}^{e, \lambda}(y, y)$ can possibly be negative but $\bar{\nu}^{e, \lambda}(y + 1, y)$ cannot.

The formal computations in the proof of Proposition 1 still work and show that $\bar{\nu}^{e, \lambda}$ has respective first and second marginals ν^e and ν^λ . Therefore

$$\begin{aligned} & \mathbf{E}_{\sigma^e, \lambda} h_k(t) - \mathbf{E}_{\tau_1 \sigma^e, \lambda} g_k(t) \\ &= \sum_{x \in \mathcal{I}} \mathbf{E}_{\sigma^e, \lambda} [h_k(t) \mid \omega_0(0) = x] \cdot \nu^e(x) - \sum_{y \in \mathcal{I}} \mathbf{E}_{\tau_1 \sigma^e, \lambda} [g_k(t) \mid \eta_0(0) = y] \cdot \nu^e(y) \\ &= \sum_{x, y \in \mathcal{I}} \left\{ \mathbf{E}_{\sigma^e, \lambda} [h_k(t) \mid \omega_0(0) = x] - \mathbf{E}_{\tau_1 \sigma^e, \lambda} [g_k(t) \mid \eta_0(0) = y] \right\} \cdot \bar{\nu}^{e, \lambda}(x, y) \\ &= \sum_{y \in \mathcal{I}} \left\{ \mathbf{E}_{\sigma^e, \lambda} [h_k(t) \mid \omega_0(0) = y] - \mathbf{E}_{\tau_1 \sigma^e, \lambda} [g_k(t) \mid \eta_0(0) = y] \right\} \cdot \bar{\nu}^{e, \lambda}(y, y) \\ &\quad + \sum_{y \in \mathcal{I}} \left\{ \mathbf{E}_{\sigma^e, \lambda} [h_k(t) \mid \omega_0(0) = y + 1] - \mathbf{E}_{\tau_1 \sigma^e, \lambda} [g_k(t) \mid \eta_0(0) = y] \right\} \cdot \bar{\nu}^{e, \lambda}(y + 1, y). \end{aligned}$$

In the first to last line the conditional initial distributions agree for ω and η , hence so do the expectations and we get zero. In the last line the mass $\bar{\nu}^{e, \lambda}(y + 1, y)$ is non-negative, and we replace this mass via (4.1):

$$\begin{aligned} & \mathbf{E}_{\sigma^e, \lambda} h_k(t) - \mathbf{E}_{\tau_1 \sigma^e, \lambda} g_k(t) \\ &= \sum_{y \in \mathcal{I}} \left\{ \mathbf{E}_{\sigma^e, \lambda} [h_k(t) \mid \omega_0(0) = y + 1] - \mathbf{E}_{\tau_1 \sigma^e, \lambda} [g_k(t) \mid \eta_0(0) = y] \right\} \cdot \bar{\nu}^{e, \lambda}(y + 1, y) \\ &= \sum_{y \in \mathcal{I}} \left\{ \mathbf{E}_{\sigma^e, \lambda} [h_k(t) \mid \omega_0(0) = y + 1] - \mathbf{E}_{\tau_1 \sigma^e, \lambda} [g_k(t) \mid \eta_0(0) = y] \right\} \cdot \hat{\nu}^{e, \lambda}(y + 1, y) \cdot (\varrho - \lambda). \end{aligned}$$

As $\hat{\nu}^{e, \lambda}$ is a proper probability distribution, at this moment we can reunify the conditional expectations at the last display to obtain

$$\begin{aligned} & \mathbf{E}_{\sigma^e, \lambda} h_k(t) - \mathbf{E}_{\tau_1 \sigma^e, \lambda} g_k(t) \\ &= \sum_{y \in \mathcal{I}} \hat{\mathbf{E}}[\hat{h}_k(t) - \hat{g}_k(t) \mid \hat{\omega}_0(0) = y + 1, \hat{\eta}_0(0) = y] \cdot \hat{\nu}^{e, \lambda}(y + 1, y) \cdot (\varrho - \lambda) \\ &= (\varrho - \lambda) \cdot \hat{\mathbf{E}}[\hat{h}_k(t) - \hat{g}_k(t)] \end{aligned}$$

with $\hat{\mathbf{E}}$ denoting the measure with initial distribution $\hat{\mu}^{e, \lambda}$ of (4.3) and following the basic coupling for evolution. Finally, notice that under the basic coupling with only one initial second class particle at the origin we have $\hat{h}_k(t) - \hat{g}_k(t) = \mathbf{1}\{Q(t) > n\}$ a.s., thus

$$\mathbf{E}_{\sigma^e, \lambda} h_k(t) - \mathbf{E}_{\tau_1 \sigma^e, \lambda} g_k(t) = (\varrho - \lambda) \cdot \hat{\mathbf{P}}\{Q(t) > n\}. \quad (7.7)$$

Next, the second way of computing (7.5). We notice that starting ω in distribution $\sigma^{\varrho, \lambda}$, letting it evolve according to its dynamics (2.1), and then defining

$$\eta_i(s) := \omega_{i+1}(s) \quad \text{for all } i \in \mathbb{Z} \text{ and } 0 \leq s \leq t$$

gives another joint realization with an implied expectation and with the corresponding joint initial measure in which the marginal distributions of ω and η are the same as before. Notice that the height functions in this coupling satisfy

$$\begin{aligned} g_k(t) &= g_k(t) - g_{-\frac{1}{2}}(t) + g_{-\frac{1}{2}}(t) - g_{-\frac{1}{2}}(0) + g_{-\frac{1}{2}}(0) \\ &= \begin{cases} -\sum_{i=0}^{k-\frac{1}{2}} \eta_i(t), & \text{if } k > -\frac{1}{2}, \\ 0, & \text{if } k = -\frac{1}{2}, \\ \sum_{i=k+\frac{1}{2}}^{-1} \eta_i(t), & \text{if } k < -\frac{1}{2} \end{cases} + g_{-\frac{1}{2}}(t) - g_{-\frac{1}{2}}(0) + g_{\frac{1}{2}}(0) + \eta_0(0) \\ &= \begin{cases} -\sum_{i=1}^{k+\frac{1}{2}} \omega_i(t), & \text{if } k > -\frac{1}{2}, \\ 0, & \text{if } k = -\frac{1}{2}, \\ \sum_{i=k+\frac{3}{2}}^0 \omega_i(t), & \text{if } k < -\frac{1}{2} \end{cases} + h_{\frac{1}{2}}(t) - h_{\frac{1}{2}}(0) + 0 + \omega_1(0) \\ &= h_{k+1}(t) - h_{\frac{1}{2}}(t) + h_{\frac{1}{2}}(t) - h_{\frac{1}{2}}(0) + \omega_1(0) \\ &= h_k(t) - \omega_{k+\frac{1}{2}}(t) + \omega_1(0), \end{aligned}$$

where we used (7.3), $h_{\frac{1}{2}}(0) = g_{\frac{1}{2}}(0) = 0$, and the fact that the heights $g_{-\frac{1}{2}}$ and $h_{\frac{1}{2}}$ change at the same time under this coupling. Thus

$$\mathbf{E}_{\sigma^{\varrho, \lambda}} h_k(t) - \mathbf{E}_{\tau_1 \sigma^{\varrho, \lambda}} g_k(t) = \mathbf{E}_{\sigma^{\varrho, \lambda}} \omega_{n+1}(t) - \mathbf{E}_{\sigma^{\varrho, \lambda}} \omega_1(0) = \mathbf{E}_{\sigma^{\varrho, \lambda}} \omega_{n+1}(t) - \lambda.$$

Together with (7.7) we arrive to the desired identity

$$\hat{\mathbf{P}}\{Q(t) > n\} = \frac{\mathbf{E}_{\sigma^{\varrho, \lambda}} \omega_{n+1}(t) - \lambda}{\varrho - \lambda},$$

which finishes the proof. ■

Proof of Theorem 2. We prove formula (5.3) in a somewhat similar fashion as we did for (5.2) before. But instead of using again the height functions we are going to work with the occupation numbers directly. Let ω be a process starting from $\sigma^{\varrho, \lambda}$ and evolving according to (2.2). Then fix $t \geq 0$, recall (5.1) and that for some $\varphi : \mathcal{I} \rightarrow \mathbb{R}$ and $x \in \mathcal{I}$: $\Phi(x) = \sum_{y=\omega^{\min}}^x \varphi(y)$, and define

$$A_n := \mathbf{E}_{\sigma^{\varrho, \lambda}} [\Phi(\omega_n(t))] - \mathbf{E}_{\sigma^{\varrho, \lambda}} [\Phi(\omega_{n+1}(t))]$$

for every $n \in \mathbb{Z}$. Now, by translation invariance we have

$$A_n = \mathbf{E}_{\sigma^{\varrho, \lambda}} [\Phi(\hat{\omega}_n(t))] - \mathbf{E}_{\tau_1 \sigma^{\varrho, \lambda}} [\Phi(\hat{\eta}_n(t))]$$

in which η evolves according to the same dynamics (2.2) as ω but starts from $\tau_1 \sigma^{\varrho, \lambda}$ instead (see (7.4)). For this latter quantity one can apply the same coupling technique we worked out in the previous proof. It then follows that

$$A_n = \sum_{x \in \mathcal{I}} \mathbf{E}_{\sigma^{\varrho, \lambda}} [\Phi(\omega_n(t)) | \omega_0(0) = x] \cdot \nu^{\varrho}(x) - \sum_{y \in \mathcal{I}} \mathbf{E}_{\tau_1 \sigma^{\varrho, \lambda}} [\Phi(\eta_n(t)) | \eta_0(0) = y] \cdot \nu^{\lambda}(y)$$

$$\begin{aligned}
&= \sum_{x,y \in \mathcal{I}} \left\{ \mathbf{E}_{\sigma^{\varrho, \lambda}} [\Phi(\omega_n(t)) \mid \omega_0(0) = x] - \mathbf{E}_{\tau_1 \sigma^{\varrho, \lambda}} [\Phi(\eta_n(t)) \mid \eta_0(0) = y] \right\} \cdot \bar{\nu}^{\varrho, \lambda}(x, y) \\
&= \sum_{y \in \mathcal{I}} \left\{ \mathbf{E}_{\sigma^{\varrho, \lambda}} [\Phi(\omega_n(t)) \mid \omega_0(0) = y] - \mathbf{E}_{\tau_1 \sigma^{\varrho, \lambda}} [\Phi(\eta_n(t)) \mid \eta_0(0) = y] \right\} \cdot \bar{\nu}^{\varrho, \lambda}(y, y) \\
&\quad + \sum_{y \in \mathcal{I}} \left\{ \mathbf{E}_{\sigma^{\varrho, \lambda}} [\Phi(\omega_n(t)) \mid \omega_0(0) = y + 1] \right. \\
&\quad \left. - \mathbf{E}_{\tau_1 \sigma^{\varrho, \lambda}} [\Phi(\eta_n(t)) \mid \eta_0(0) = y] \right\} \cdot \bar{\nu}^{\varrho, \lambda}(y + 1, y),
\end{aligned}$$

where we used that the joint measure $\bar{\nu}^{\varrho, \lambda}$, defined in (7.6), has respective marginals ν^{ϱ} and ν^{λ} . As before we can take advantage of the fact that the last but one sum in the previous display vanishes and that $\bar{\nu}^{\varrho, \lambda}(y + 1, y) = \hat{\nu}^{\varrho, \lambda}(y + 1, y) \cdot (\varrho - \lambda)$ so

$$\begin{aligned}
A_n &= \sum_{y \in \mathcal{I}} \hat{\mathbf{E}}[\Phi(\hat{\omega}_n(t)) - \Phi(\hat{\eta}_n(t)) \mid \hat{\omega}_0(0) = y + 1, \hat{\eta}_0(0) = y] \cdot \hat{\nu}^{\varrho, \lambda}(y + 1, y) \cdot (\varrho - \lambda) \\
&= (\varrho - \lambda) \cdot \hat{\mathbf{E}}[\mathbb{1}\{Q(t) = n\} \cdot \varphi(\hat{\omega}_n(t))]
\end{aligned}$$

using again that $\hat{\nu}$ is a proper probability distribution and

$$\Phi(\hat{\omega}_n(t)) - \Phi(\hat{\eta}_n(t)) = \mathbb{1}\{Q(t) = n\} \cdot \varphi(\hat{\omega}_n(t))$$

a.s. under the basic coupling.

Thus we obtained the identity $\hat{\mathbf{E}}[\mathbb{1}\{Q(t) = n\} \cdot \varphi(\hat{\omega}_{Q(t)}(t))] = \frac{1}{\varrho - \lambda} \cdot A_n$ for every $n \in \mathbb{Z}$. Now, summing these equations up as n runs from $-L$ to L for some $L \in \mathbb{Z}^+$, we arrive to

$$\hat{\mathbf{E}}[\mathbb{1}\{Q(t) \in [-L, L]\} \cdot \varphi(\hat{\omega}_{Q(t)}(t))] = \frac{1}{\varrho - \lambda} \cdot (\mathbf{E}_{\sigma^{\varrho, \lambda}} \Phi(\omega_{-L}(t)) - \mathbf{E}_{\sigma^{\varrho, \lambda}} \Phi(\omega_{L+1}(t))). \quad (7.8)$$

Notice that $\mathbf{E}_{\sigma^{\varrho, \lambda}} \Phi(\omega_{\pm L}(t))$ equals to $\mathbf{E}_{\tau_{\pm L} \sigma^{\varrho, \lambda}} \Phi(\omega_0(t))$ for all $L \in \mathbb{Z}^+$ due to the translation invariant property of the rates.

First, assume that $\varphi \geq 0$ and $\mathbf{E}_{\sigma^{\varrho, \varrho}} \Phi(\omega_0(t)) < +\infty$ hold. Since the measures $\tau_{\pm L} \sigma^{\varrho, \lambda}$ are dominated by $\sigma^{\varrho, \varrho}$ for every $L \in \mathbb{N}_0$ and Φ is monotone non-decreasing, it then follows by attractiveness that each of the above expectations of $\Phi(\omega_0(t))$ is finite. Let M be a positive real and define Φ_M to be $\Phi \wedge M$. It is easy to see that Φ_M and $\Phi - \Phi_M$ are non-negative and monotone non-decreasing functions as well. Hence

$$\begin{aligned}
&|\mathbf{E}_{\tau_{\pm L} \sigma^{\varrho, \lambda}} \Phi(\omega_0(t)) - \mathbf{E}_{\sigma^{\varrho, \varrho}} \Phi(\omega_0(t))| \\
&\leq |\mathbf{E}_{\tau_{\pm L} \sigma^{\varrho, \lambda}} \Phi_M(\omega_0(t)) - \mathbf{E}_{\sigma^{\varrho, \varrho}} \Phi_M(\omega_0(t))| + 2 \cdot \mathbf{E}_{\sigma^{\varrho, \varrho}} (\Phi - \Phi_M)(\omega_0(t)),
\end{aligned}$$

using again the attractiveness of ω . At this point M can be made large enough for the last quantity to be small, independently of L . Also, note that the measures $\tau_L \sigma^{\varrho, \lambda}$ converge weakly to $\sigma^{\varrho, \varrho}$ and to $\sigma^{\lambda, \lambda}$ as $L \rightarrow \pm\infty$, respectively. Now, due to a finite speed of propagation of information (implied by construction of the dynamics), and taking advantage of that Φ_M is bounded, it follows that $\lim_{L \rightarrow +\infty} \mathbf{E}_{\tau_L \sigma^{\varrho, \lambda}} \Phi_M(\omega_0(t)) = \mathbf{E}_{\sigma^{\varrho, \varrho}} \Phi_M(\omega_0(t))$ and $\lim_{L \rightarrow -\infty} \mathbf{E}_{\tau_L \sigma^{\varrho, \lambda}} \Phi_M(\omega_0(t)) = \mathbf{E}_{\sigma^{\lambda, \lambda}} \Phi_M(\omega_0(t))$. What we have just proved implies that the right-hand side of (7.8) is bounded hence the left-hand side converges to $\hat{\mathbf{E}} \varphi(\hat{\omega}_{Q(t)}(t))$ by monotone convergence, and so we get the desired formula (5.3).

On the other hand, assume that φ is absolutely summable, that is $\sum_{y \in \mathcal{I}} |\varphi(y)| < +\infty$. It follows that φ as well as Φ are bounded functions. By dominated convergence the left-hand side of (7.8) converges to $\hat{\mathbf{E}} \varphi(\hat{\omega}_{Q(t)}(t))$ as $L \rightarrow +\infty$, while the convergence of the right-hand side follows directly from finite information propagation velocity and the weak limit of $\tau_{\pm L} \sigma^{\varrho, \lambda}$ as $L \rightarrow +\infty$.

We remark that the way we obtained (5.3) could have been used to obtain (5.2) as well. For historical reasons in the previous case we followed the approach that was inherited from [17]. ■

Proof of Proposition 4. Basically, we will follow the lines of [25, pp. 1791–1792]. Fix a $t \in \mathbb{R}^+$ and let $x \in \mathbb{R}$ be a continuity point of $u^0(\cdot, t)$. Furthermore, let $\varphi : \Omega \rightarrow \mathbb{R}$ be a monotone non-decreasing cylinder function for which (5.11) holds. Note that the measure $\tau_j \sigma^{\varrho, \lambda}$ is stochastically

dominated by $\tau_k \sigma^{\varepsilon, \lambda}$ for every $j \geq k$. Then by attractivity we have $\mathbf{E}_{\tau_{[N\varepsilon]} \sigma^{\varepsilon, \lambda}} \varphi(\tau_k \omega(Nt)) \leq \mathbf{E}_{\sigma^{\varepsilon, \lambda}} \varphi(\tau_{[Nx]} \omega(Nt)) \leq \mathbf{E}_{\tau_{[N\varepsilon]} \sigma^{\varepsilon, \lambda}} \varphi(\tau_j \omega(Nt))$ for every $j \geq [Nx] - [N\varepsilon] \geq k$, where $\varepsilon \in \mathbb{R}$. Now, let $\varepsilon_+ > 0$ and $\varepsilon_- < 0$. Then

$$\begin{aligned} & \frac{1}{|[N\varepsilon_-]| + 1} \sum_{k: [Nx] - |[N\varepsilon_-]| \leq k \leq [Nx]} \mathbf{E}_{\tau_{[N\varepsilon_-]} \sigma^{\varepsilon, \lambda}} \varphi(\tau_j \omega(Nt)) \\ & \leq \mathbf{E}_{\sigma^{\varepsilon, \lambda}} \varphi(\tau_{[Nx]} \omega(Nt)) \leq \frac{1}{|[N\varepsilon_+]| + 1} \sum_{j: [Nx] \leq j \leq [Nx] + [N\varepsilon_+]} \mathbf{E}_{\tau_{[N\varepsilon_+]} \sigma^{\varepsilon, \lambda}} \varphi(\tau_j \omega(Nt)). \end{aligned}$$

Taking the limit inferior then the superior as $N \rightarrow +\infty$ and using assumption (5.11) we obtain that

$$\begin{aligned} & \frac{1}{|\varepsilon_-|} \int_{z: x - |\varepsilon_-| \leq z \leq x} \mathbf{E}_{\pi^{u^{\varepsilon_-}(z, t)}} \varphi(\omega) dz \\ & \leq \liminf_{N \rightarrow +\infty} \mathbf{E}_{\sigma^{\varepsilon, \lambda}} \varphi(\tau_{[Nx]} \omega(Nt)) \leq \limsup_{N \rightarrow +\infty} \mathbf{E}_{\sigma^{\varepsilon, \lambda}} \varphi(\tau_{[Nx]} \omega(Nt)) \\ & \leq \frac{1}{\varepsilon_+} \int_{y: x \leq y \leq x + \varepsilon_+} \mathbf{E}_{\pi^{u^{\varepsilon_+}(y, t)}} \varphi(\omega) dy. \end{aligned}$$

Since $(\pi^\varepsilon)_{\varepsilon \in \mathbb{R}}$ is an ordered family of measures and u^{ε_-} as well as u^{ε_+} are monotone non-increasing functions it follows that

$$\begin{aligned} & \mathbf{E}_{\pi^{u^{\varepsilon_-}(x, t)}} \varphi(\omega) \\ & \leq \liminf_{N \rightarrow +\infty} \mathbf{E}_{\sigma^{\varepsilon, \lambda}} \varphi(\tau_{[Nx]} \omega(Nt)) \leq \limsup_{N \rightarrow +\infty} \mathbf{E}_{\sigma^{\varepsilon, \lambda}} \varphi(\tau_{[Nx]} \omega(Nt)) \\ & \leq \mathbf{E}_{\pi^{u^{\varepsilon_+}(x, t)}} \varphi(\omega). \end{aligned}$$

Without loss of generality we can assume that $\varphi \geq 0$. Now, let $M \in \mathbb{R}^+$ and define $\varphi_M = \varphi \wedge M$. Then

$$\begin{aligned} & |\mathbf{E}_{\pi^{u^{\varepsilon_\pm}(x, t)}} \varphi(\omega) - \mathbf{E}_{\pi^{u^0(x, t)}} \varphi(\omega)| \\ & \leq |\mathbf{E}_{\pi^{u^{\varepsilon_\pm}(x, t)}} \varphi_M(\omega) - \mathbf{E}_{\pi^{u^0(x, t)}} \varphi_M(\omega)| + 2 \mathbf{E}_{\pi^{\varepsilon_{\max}}} (\varphi - \varphi_M)(\omega) \end{aligned}$$

by stochastic dominance of the measures $(\pi^\varepsilon)_{\varepsilon \in \mathbb{R}}$. Note that the last quantity in the previous display can be made arbitrarily small by choosing M to be large enough while the first one vanishes as $\varepsilon \rightarrow 0$ by weak convergence. This results in the desired limit (5.12).

Since any bounded cylinder function can be written as the difference of two monotone cylinder functions we have finished the proof. \blacksquare

Proof of Theorem 6 Under the assumptions of Theorem 4 we have in hand both the set of product-form extremal stationary distributions and the convergence result (5.10) (see also [30, Section 7]). Hence we can apply Proposition 4 with $\varphi(\omega) = \omega_0$ and the desired limit can be obtained via (5.2).

On the other hand, when the setup of Theorem 5 is considered, we can take advantage of the bounded occupation numbers. But first let us mention that in general, the density parameterized ergodic set of stationary distributions have been proved to exist and they form a continuous and stochastically ordered family of measures with a closed parameter set \mathcal{R} of $[0, K]$ containing 0 and K (see [3, Section 3, Proposition 3.1]). So we only need to take care of the limit in (5.11) when $\varphi(\omega) = \omega_0$. Without loss of generality one can assume that $\omega^{\min} \geq 0$ and that the compact support of ψ has (at most) unit Lebesgue measure. Then we have

$$\begin{aligned} & \mathbf{E}_{\tau_{[N\varepsilon]} \sigma^{\varepsilon, \lambda}} \left| \frac{1}{N} \sum_{j \in \mathbb{Z}} \psi(j/N) \cdot \omega_j^{\varepsilon, N}(Nt) - \int_{x \in \mathbb{R}} \psi(x) \cdot u^\varepsilon(x, t) dx \right| \\ & \leq \varepsilon + \omega^{\max} \cdot \max_{x \in \mathbb{R}} |\psi(x)| \cdot \mathbf{P}^N \left(\left| \frac{1}{N} \sum_{j \in \mathbb{Z}} \psi(j/N) \cdot \omega_j^{\varepsilon, N}(Nt) - \int_{x \in \mathbb{R}} \psi(x) \cdot u^\varepsilon(x, t) dx \right| > \varepsilon \right), \end{aligned}$$

where $\varepsilon > 0$ can be chosen arbitrarily. Now, taking the limit as $N \rightarrow +\infty$ the desired formula easily follows. Finally, we can conclude using (5.2) again.

In both cases the monotonicity, boundedness and convergence of the entropy solutions $(u^\varepsilon)_{\varepsilon \in \mathbb{R}}$ come from the classical results of hyperbolic conservation laws (see [23] and further references therein). \blacksquare

Proof of Theorem 7 By shifting the whole system upwards we can assume, without loss of generality, that $\omega^{\min} = 0$. Now, we are going to handle both the finite and infinite settings. Attractivity and the fact $\omega_n(t) \geq 0$ imply that $\sup_{n \in \mathbb{Z}} \mathbf{E}_{\sigma^{\varepsilon, \lambda}} \omega_n(t)^2$ can be estimated from above by $\mathbf{E}_{\sigma^{\varepsilon, \ell}} \omega_0(t)^2$ which is supposed to be finite. It follows that for each fixed $t \geq 0$, the sequence

$$\left(\frac{1}{N} \sum_{j \in \mathbb{Z}} \psi(j/N) \cdot \omega_j^{\varepsilon, N}(N^2 t) \right)_{N \in \mathbb{Z}^+}$$

is bounded in \mathcal{L}^2 for every $\varepsilon \in \mathbb{R}$, where $\omega^{\varepsilon, N}$ starts from $\tau_{[N\varepsilon]} \sigma^{\varepsilon, \lambda}$ and ψ is a given continuous function of compact support. Hence, the conservation of local equilibrium we assumed implies (5.11) for $\varphi(\omega) = \omega_0$. Since the time scaling played no role in (the proof of) Proposition 4, one can save this result to the diffusive case as well, resulting in the desired convergence (5.16). \blacksquare

Proof of Theorem 8. Without loss of generality we can assume that $\omega^{\min} = 0$ while ω^{\max} is some fixed positive integer. We start the process $\hat{\eta}$ from initial configuration $\hat{\eta}_0$, where $\hat{\eta}_0 = \omega^{\max} \mathbf{1}_{\{i \leq 0\}}$. For sake of simplicity we let $\bar{r} := r_0(\hat{\eta}_0)$, which is positive by the non-degeneracy of the rate function r . In what follows, we will work with the height function \mathbf{g} of $\hat{\eta}$ (see its definition in (7.3)).

Since the initial configuration $\hat{\eta}_0$ of $\hat{\eta}$ can change only due to a jump performed by a particle from the origin to lattice point 1, it follows that in a small time interval $[0, \varepsilon]$ we should only take into account two events: $\hat{\eta}(\varepsilon) = \hat{\eta}_0$ occurring with probability $1 - \varepsilon \cdot \bar{r}$ (up to first order in ε) and $\hat{\eta}(\varepsilon) = \hat{\eta}_0 - \delta_0 + \delta_1$ which in turn occurs with rate $\varepsilon \cdot \bar{r}$. All the other moves are of order $o(\varepsilon)$ in probability. Putting the above together we arrive to

$$\begin{aligned} & \frac{d}{dt} \mathbf{E}_{\hat{\eta}_0} g_{\frac{1}{2}}(t) \\ &= \lim_{\varepsilon \downarrow 0} \frac{\mathbf{E}_{\hat{\eta}_0} g_{\frac{1}{2}}(t + \varepsilon) - \mathbf{E}_{\hat{\eta}_0} g_{\frac{1}{2}}(t)}{\varepsilon} \\ &= \bar{r} \cdot \lim_{\varepsilon \downarrow 0} \left(\mathbf{E}_{\hat{\eta}_0} [\mathbf{E}[g_{\frac{1}{2}}(t + \varepsilon) \mid \hat{\eta}(\varepsilon) = \hat{\eta}_0 - \delta_0 + \delta_1] - \mathbf{E}[g_{\frac{1}{2}}(t + \varepsilon) \mid \hat{\eta}(\varepsilon) = \hat{\eta}_0]] \right) \\ &= \bar{r} \cdot \left(1 + \mathbf{E}_{\hat{\omega}_0} h_{\frac{1}{2}}(t) - \mathbf{E}_{\hat{\eta}_0} g_{\frac{1}{2}}(t) \right) \end{aligned} \quad (7.9)$$

by the Markov property, where \mathbf{h} is the height function of $\hat{\omega}$ that starts from $\hat{\omega}_0 = \hat{\eta}_0 - \delta_0 + \delta_1$. Along the way we have also used the fact that $g_{\frac{1}{2}}(t)$ counts exactly how many (signed) particle jumps occurred above the bond $[0, 1]$ until time $t > 0$. Also recall that $g_{\frac{1}{2}}(0)$ (and $h_{\frac{1}{2}}(0)$) is set to be zero by choice. ((7.9) is also known as the *Kolmogorov forward equation*.)

We now couple the processes $\hat{\omega}$ and $\hat{\eta}$ coordinate-wise, employing the basic coupling with deterministic initial configurations $\hat{\omega}_0$ and $\hat{\eta}_0$, respectively. Notice that there are two second class particles in the system $(\hat{\omega}, \hat{\eta})$: a negative and a positive starting from positions 0 and 1, respectively. It then follows that under this coupling

$$\begin{aligned} 1 + h_{\frac{1}{2}}(t) - g_{\frac{1}{2}}(t) &= \sum_{j=1}^{+\infty} (\hat{\omega}_j(t) - \hat{\eta}_j(t)) = \sum_{j=1}^{+\infty} s_j(t) \cdot n_j(t) \\ &\leq \mathbf{1}\{\mathcal{N}(s) = 2 \text{ for all } 0 \leq s \leq t\} \end{aligned} \quad (7.10)$$

holds a.s. On the other hand

$$\begin{aligned} \frac{d}{dt} \mathbf{E}_{\hat{\eta}_0} g_{\frac{1}{2}}(t) &= \lim_{\varepsilon \downarrow 0} \frac{\mathbf{E}_{\hat{\eta}_0} [g_{\frac{1}{2}}(t + \varepsilon) - g_{\frac{1}{2}}(t)]}{\varepsilon} \\ &= \lim_{\varepsilon \downarrow 0} \frac{\mathbf{E}_{\hat{\eta}_0} [\varepsilon \cdot r_0(\hat{\eta}(t)) \cdot \mathbf{E}[1 \mid \hat{\eta}(t)] - \varepsilon \cdot \ell_0(\hat{\eta}(t)) \cdot \mathbf{E}[1 \mid \hat{\eta}(t)]]}{\varepsilon} \end{aligned}$$

$$= \mathbf{E}_{\hat{\eta}_0}[r_0(\hat{\eta}(t)) - \ell_0(\hat{\eta}(t))], \quad (7.11)$$

using again the Markov property and that $g_{\frac{1}{2}}$ can change only if a particle attempts to jump either from 0 to 1 or from 1 to 0. ((7.11) is sometimes called as the *Kolmogorov backward equation*.)

Putting (7.9) and (7.11) together and using the estimate from (7.10) we arrive to

$$\frac{1}{r} \mathbf{E}_{\hat{\eta}_0}[r_0(\hat{\eta}(t)) - \ell_0(\hat{\eta}(t))] \leq \mathbf{P}\{\mathcal{N}(s) = 2 \text{ for all } 0 \leq s \leq t\}.$$

Now, taking the limit superior as $t \rightarrow +\infty$ we obtain (5.17) by monotone convergence.

In the totally asymmetric case, it is easy to see that there must be a subsequence $(t_m)_{m=1}^{+\infty}$ for which

$$\lim_{m \rightarrow +\infty} \mathbf{E}_{\hat{\eta}_0} r_0(\hat{\eta}(t_m)) > 0,$$

since otherwise we would have $\lim_{t \rightarrow +\infty} p_0(\hat{\eta}(t)) = 0$ a.s. by dominated convergence, implying that the probability of the event $\{\hat{\eta}_0(t) = 0 \text{ or } \hat{\eta}_1(t) = \omega^{\max}\}$ tends to 1 as $t \rightarrow +\infty$. But this obviously cannot happen.

Finally, for those models that satisfy the (further) conditions of Theorem 3 we can apply Theorem 4 and then Proposition 4, which completes the proof by choosing the cylinder function φ to be $r_0 - \ell_0$ in (5.10). ■

References

- [1] G. Amir, O. Angel, and B. Valkó. The TASEP speed process. *Ann. Probab.*, 39(4):1205–1242, 2011.
- [2] E. D. Andjel. Invariant measures for the zero range process. *Ann. Probab.*, 10:525–547, 1982.
- [3] C. Bahadoran, H. Guiol, K. Ravishankar, and E. Saada. Euler hydrodynamics of one-dimensional attractive particle systems. *Ann. Probab.*, 34(4):1339–1369, 2006.
- [4] C. Bahadoran, H. Guiol, K. Ravishankar, and E. Saada. Strong hydrodynamic limit for attractive particle systems on \mathbb{Z} . *Electron. J. Probab.*, 15(1):1–43, 2010.
- [5] M. Balázs. Microscopic shape of shocks in a domain growth model. *J. Stat. Phys.*, 105(3-4):511–524, 2001.
- [6] M. Balázs. Growth fluctuations in a class of deposition models. *Ann. Inst. H. Poincaré Probab. Statist.*, 39(4):639–685, 2003.
- [7] M. Balázs. Multiple shocks in bricklayers’ model. *J. Stat. Phys.*, 117(1-2):77–98, 2004.
- [8] M. Balázs, Gy. Farkas, P. Kovács, and A. Rákos. Random walk of second class particles in product shock measures. *J. Stat. Phys.*, 139(2):252–279, 2010.
- [9] M. Balázs, A. L. Nagy, B. Tóth, and I. Tóth. Coexistence of shocks and rarefaction fans: complex phase diagram of a simple hyperbolic particle system. Submitted manuscript, <http://arxiv.org/abs/1601.02161>, 2016.
- [10] M. Balázs, F. Rassoul-Agha, T. Seppäläinen, and S. Sethuraman. Existence of the zero range process and a deposition model with superlinear growth rates. *Ann. Probab.*, 35(4):1201–1249, 2007.
- [11] M. Balázs and T. Seppäläinen. A convexity property of expectations under exponential weights. Preprint, available at <http://arxiv.org/pdf/0707.4273v2.pdf>, 2007.
- [12] M. Balázs and T. Seppäläinen. Exact connections between current fluctuations and the second class particle in a class of deposition models. *J. Stat. Phys.*, 127(2):431–455, 2007.
- [13] C. Coccozza-Thivent. Processus des misanthropes. *Z. Wahrsch. Verw. Gebiete*, 70(4):509–523, 1985.

- [14] B. Derrida, J. L. Lebowitz, and E. R. Speer. Shock profiles for the asymmetric simple exclusion process in one dimension. *J. Stat. Phys.*, 89(1-2):135–167, 1997.
- [15] P. A. Ferrari. Shocks in one-dimensional processes with drift. In G. Grimmett, editor, *Probability and Phase Transition*, pages 35–48. Kluwer Academic Publishers, The Netherlands, 1994.
- [16] P. A. Ferrari, P. Gonçalves, and J. B. Martin. Collision probabilities in the rarefaction fan of asymmetric exclusion processes. *Ann. Inst. H. Poincaré Probab. Statist.*, 45(4):1048–1064, 2009.
- [17] P. A. Ferrari and C. Kipnis. Second class particles in the rarefaction fan. *Ann. Inst. H. Poincaré Probab. Statist.*, 31(1):143–154, 1995.
- [18] P. A. Ferrari, J. B. Martin, and L. P. R. Pimentel. A phase transition for competition interfaces. *Ann. Appl. Probab.*, 19(1):281–317, 2009.
- [19] P. A. Ferrari and L. P. R. Pimentel. Competition interfaces and second class particles. *Ann. Probab.*, 33(4):1235–1254, 2005.
- [20] P. A. Ferrari, E. Presutti, and M. E. Vares. Local equilibrium for a one dimensional zero range process. *Stoch. Proc. Appl.*, 26:31–45, 1987.
- [21] P. Gonçalves. On the asymmetric zero-range in the rarefaction fan. *J. Stat. Phys.*, 154(4):1074–1095, 2014.
- [22] H. Guiol and T. Mountford. Questions for second class particles in exclusion processes. *Markov Process. Rel. Fields*, 12(2):301–308, 2006.
- [23] H. Holden and N. H. Risebro. *Front Tracking for Hyperbolic Conservation Laws*. Springer, 2011.
- [24] C. Kipnis and C. Landim. *Scaling Limits of Interacting Particle Systems*. Springer, 1999.
- [25] C. Landim. Conservation of local equilibrium for attractive particle systems on \mathbb{Z}^d . *Ann. Probab.*, 21(4):1782–1808, 1993.
- [26] C. Landim and M. Mourragui. Hydrodynamic limit of mean zero asymmetric zero range processes in infinite volume. *Ann. Inst. H. Poincaré Probab. Statist.*, 33(1):65–82, 1997.
- [27] T. M. Liggett. *Interacting Particle Systems*. Springer, 1985.
- [28] T. Mountford and H. Guiol. The motion of a second class particle for the tasep starting from a decreasing shock profile. *Ann. Appl. Probab.*, 15(2):1227–1259, 2005.
- [29] A. Perrut. Hydrodynamic limit for a nongradient system in infinite volume. *Stoch. Proc. Appl.*, 84(2):227–253, 1999.
- [30] F. Rezakhanlou. Hydrodynamic limit for attractive particle systems on \mathbb{Z}^d . *Comm. Math. Phys.*, 140(3):417–448, 1991.
- [31] F. Rezakhanlou. Microscopic structure of shocks in one conservation laws. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 12(2):119–153, 1995.
- [32] D. Romik and P. Śniady. Jeu de taquin dynamics on infinite Young tableaux and second class particles. *Ann. Probab.*, 43(2):682–737, 2015.
- [33] T. Seppäläinen. Existence of hydrodynamics for the totally asymmetric simple k -exclusion process. *Ann. Probab.*, 27(1):361–415, 1999.
- [34] T. Seppäläinen. Translation Invariant Exclusion Processes. Unpublished book, available at <http://www.math.wisc.edu/~seppalai/excl-book/ajo.pdf>, 2008.
- [35] B. Tóth and B. Valkó. Between equilibrium fluctuations and Eulerian scaling: perturbation of equilibrium for a class of deposition models. *J. Stat. Phys.*, 109(1):177–205, 2002.
- [36] C. A. Tracy and H. Widom. On the distribution of a second-class particle in the asymmetric simple exclusion process. *J. Phys. A: Math. Theor.*, 42(42):425002, 2009.